HW 10

(1.) $3u_x + 5u_t = 0$, IC: $u(x,0) = f(x)$

Fourier transform the PDE: $\quad u = u(x,t)$, $\hat{u} = \hat{u}(\xi,t)$

$$0 = \widehat{3u_x + 5u_t} = 3\,\widehat{u_\xi} + 5\,\hat{u}_t = 3i\xi\hat{u} + 5\hat{u}_t$$

$\Leftrightarrow \quad \hat{u}_t + \frac{3}{5}i\xi\hat{u} = 0$ is an ODE in t with solution $\hat{u}(\xi,t) = c(\xi)e^{-\frac{3}{5}i\xi t}$

use IC: $u(x,0) = f(x) \Rightarrow \hat{u}(\xi,0) = \hat{f}(\xi)$, so

$$\hat{f}(\xi) = \hat{u}(\xi,0) = c(\xi)$$

it follows that $\hat{u}(\xi,t) = \hat{f}(\xi)e^{-\frac{3}{5}i\xi t}$

Fourier transform back:

$$u(x,t) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \hat{f}(\xi)e^{-\frac{3}{5}i\xi t}e^{i\xi x}\,d\xi$$

$$= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \hat{f}(\xi)e^{i\xi(x-\frac{3}{5}t)}\,d\xi = f(x-\tfrac{3}{5}t)$$

So, $u(x,t) = f(x-\frac{3}{5}t)$.

(2.) $3t\,u_x + 5u_t = 0$, IC: $u(x,0) = f(x)$

Fourier transform the PDE:

$$0 = \widehat{3t\,u_x + 5u_t} = 3t\,\widehat{u_\xi} + 5\hat{u}_t = 3ti\xi\hat{u} + 5\hat{u}_t$$

$\Leftrightarrow \quad \hat{u}_t + \frac{3}{5}i\xi t\hat{u} = 0$, integrating factor: $e^{\int \frac{3}{5}i\xi t\,dt} = e^{\frac{3}{10}i\xi t^2}$

$$\Leftrightarrow \quad \underbrace{\hat{u}_t e^{\frac{3}{10}i\xi t^2} + \frac{3}{5}i\xi t\hat{u}e^{\frac{3}{10}i\xi t^2}}_{=\left(\hat{u}e^{\frac{3}{10}i\xi t^2}\right)_t} = 0$$

$\Rightarrow \quad \hat{u}e^{\frac{3}{10}i\xi t^2} = c(\xi)$

$\Leftrightarrow \quad \hat{u}(\xi,t) = c(\xi)e^{-\frac{3}{10}i\xi t^2}$

use IC: $u(x,0) = f(x) \Rightarrow \hat{u}(\xi,0) = \hat{f}(\xi)$, so $\hat{f}(\xi) = \hat{u}(\xi,0) = c(\xi)$

It follows that $\hat{u}(\xi,t) = \hat{f}(\xi)e^{-\frac{3}{10}i\xi t^2}$

Fourier transform back:

$$u(x,t) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \hat{f}(\xi)e^{-\frac{3}{10}i\xi t^2}e^{i\xi x}\,d\xi = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \hat{f}(\xi)e^{i\xi(x-\frac{3}{10}t^2)} = f(x-\tfrac{3}{10}t^2)$$

So, $u(x,t) = f(x-\frac{3}{10}t^2)$.

③ $u_t + u_x + u = 0$, IC: $u(x,0) = f(x)$

Fourier transform the PDE:

$$0 = \widehat{u_t + u_x + u} = \hat{u}_t + \widehat{u_\xi} + \hat{u} = \hat{u}_t + i\xi\hat{u} + \hat{u} = \hat{u}_t + (i\xi+1)\hat{u}$$ is an ODE in t

with solution $\hat{u}(\xi,t) = c(\xi)e^{-(i\xi+1)t}$

use IC: $u(x,0) = f(x) \Rightarrow \hat{u}(\xi,0) = \hat{f}(\xi)$,

so $\hat{f}(\xi) = \hat{u}(\xi,0) = c(\xi)$

it follows that $\hat{u}(\xi,t) = \hat{f}(\xi)e^{-(i\xi+1)t}$

Fourier transform back:

$$u(x,t) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \hat{f}(\xi)e^{-(i\xi+1)t}e^{i\xi x}\,d\xi$$

$$= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \hat{f}(\xi)e^{i\xi(x-t)}\,d\xi\, e^{-t} = e^{-t}f(x-t)$$

So, $u(x,t) = e^{-t}f(x-t)$.

This solves the PDE:

$$u_t = +e^{-t}\cdot f'(x-t)(-1) \neq \cancel{e^{-t}f'(x-t)} + (-e^{-t})f(x-t)$$

$$u_x = e^{-t}f'(x-t)$$

$$u = e^{-t}f(x-t)$$

$$\Rightarrow u_t + u_x + u = -e^{-t}f'(x-t) - e^{-t}f(x-t)$$
$$+ e^{-t}f'(x-t) + e^{-t}f(x-t) = 0 \quad ✓$$

⑤ $u_t = ku_{xx} + u$, $u(x,0) = f(x)$, $-\infty < x < \infty$, $t > 0$ (see also lecture 21)

Fourier transform the PDE:

$$\widehat{u_t} = \widehat{ku_{xx} + u} = k\widehat{u_{\xi\xi}} + \hat{u} = -k\xi^2\hat{u} + \hat{u} = (1-k\xi^2)\hat{u}$$ is an ODE in t

with solution $\hat{u}(\xi,t) = c(\xi)e^{(1-k\xi^2)t}$

use IC: $u(x,0) = f(x) \Rightarrow \hat{u}(\xi,0) = \hat{f}(\xi)$, so $\hat{f}(\xi) = \hat{u}(\xi,0) = c(\xi)$

it follows that $\hat{u}(\xi,t) = \hat{f}(\xi)e^{(1-k\xi^2)t}$

Fourier transform back:

$$u(x,t) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \hat{f}(\xi)e^{(1-k\xi^2)t}e^{i\xi x}\,d\xi$$

What g satisfies $\hat{g}(\xi) = e^{(1-k\xi^2)t}$?

$$\hat{g}(\xi) = e^{(1-k\xi^2)t} \Rightarrow g(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\hat{g}(\xi)e^{i\xi x}\,d\xi = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{(1-k\xi^2)t+i\xi x}\,d\xi$$

$$= \underbrace{\left(\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-k\xi^2 t+i\xi x}\,d\xi\right)}_{=\frac{1}{\sqrt{2kt}}e^{-\frac{x^2}{4kt}} \text{ (see lecture 21)}} e^{t}$$

$$= \frac{1}{\sqrt{2kt}} e^{t - \frac{x^2}{4kt}}$$

Now, $\hat{u}(\xi, t) = \hat{f}(\xi)\, \hat{g}(\xi)$

$$\Rightarrow u(x,t) = \frac{1}{\sqrt{2\pi}} (f * g) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(y)\, g(x-y)\, dy$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(y) \frac{1}{\sqrt{2kt}} e^{t - \frac{(x-y)^2}{4kt}} dy$$

$$= \frac{1}{2\sqrt{\pi k t}} \int_{-\infty}^{\infty} f(y) e^{t - \frac{(x-y)^2}{4kt}} dy$$

$$= \frac{1}{\sqrt{4\pi k t}} e^{t} \int_{-\infty}^{\infty} f(y)\, e^{-\frac{(x-y)^2}{4kt}} dy$$