Math 241, Fall 2015 **Renato Ghini Bettiol**

Practice problems for Midterm 1

Due: Never

1. Let $u(x,y)$ be the solution to the Laplace equation $\Delta u = 0$ on the square $0 < x < 1$, $0 < y < 1$, with boundary value $+1$ on the top and 0 on all other sides, that is, $u(x,1) = 1$ and $u(x,0) = u(0,y) = u(1,y) = 0$.

   a) Without finding an explicit formula for $u(x,y)$, compute $u\left(\frac{1}{2}, \frac{1}{2}\right)$.

   b) What are the maximum and minimum values attained by $u(x,y)$ on the (closed) square $0 \leq x \leq 1$, $0 \leq y \leq 1$?

2. Consider the heat equation with source $u_t = u_{xx} + 4$, where $0 < x < 1$, with boundary conditions given by $u_x(0,t) = 5$ and $u_x(1,t) = \beta$. Determine the value of β for which an equilibrium solution exists. Write an explicit formula for the equilibrium solution (up to a constant).

3. Consider the heat equation with source $u_t = u_{xx} + 4$, where $0 < x < 1$, with boundary conditions and initial condition given by $u_x(0,t) = 5$, $u_x(1,t) = \beta$ and $u(x,0) = f(x)$. Compute the total thermal energy $H(t) = \int_0^1 u(x,t)\,\mathrm{d}x$, in terms of β and $f(x)$.

   Hint: Compute $H'(t)$ using the PDE and boundary conditions, and then compute $H(0)$. Observe that the only value of β for which the total thermal energy $H(t)$ is constant is the value for which an equilibrium solution exists (compare with the previous problem).

4. Solve the heat equation $u_t = 3\,u_{xx}$, where $0 < x < 1$, with boundary conditions and initial conditions given by $u(0,t) = u(1,t) = 0$ and $u(x,0) = 2\sin(\pi x) + 5\sin(4\pi x)$.

5. Solve the heat equation $u_t = 3\,u_{xx}$, where $0 < x < 1$, with boundary conditions and initial conditions given by $u_x(0,t) = u_x(1,t) = 0$ and $u(x,0) = 2 + 3\cos(4\pi x)$.

6. Solve the wave equation $u_{tt} = 4\,u_{xx}$ for $0 < x < 1$, with the boundary conditions given by $u(0,t) = u(1,t) = 0$, and initial conditions given by $u(x,0) = 2\sin(3\pi x)$, $u_t(x,0) = 6\sin(9\pi x)$.

7. Solve the wave equation $u_{tt} = 4\,u_{xx}$ for $0 < x < 1$, with the boundary conditions given by $u_x(0,t) = u_x(1,t) = 0$, and initial conditions given by $u(x,0) = 1 + 2\cos(3\pi x)$, $u_t(x,0) = 2 + 6\cos(9\pi x)$.

8. Solve the wave equation $u_{tt} = 4\,u_{xx}$ for $-\infty < x < +\infty$, with initial conditions given by $u(x,0) = 2\sin(3\pi x)$ and $u_t(x,0) = 6\sin(9\pi x)$.

9. Find the solution to the Laplace equation

$$\Delta u = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} = 0$$

on the disk of radius 1, which is bounded at the origin $|u(0,\theta)| < +\infty$, and satisfies $u(1,\theta) = 7 + 9\cos(\theta)$.

10. Find the solution to the Laplace equation

$$\Delta u = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} = 0$$

on the region of the plane *outside* the disk of radius 1, which is bounded as $r \to +\infty$, and satisfies $u(1,\theta) = 7 + 9\cos(\theta)$.

11. Find the eigenvalues $\{\lambda_n\}$ and corresponding eigenfunctions $\{\phi_n(x)\}$ of the following fourth order eigenvalue problem:

$$\begin{cases} X''''(x) - \lambda X(x) = 0 \\ X(0) = X(L) = 0 \\ X''(0) = X''(L) = 0 \end{cases}$$

Hint: The roots of the equation $r^4 - \lambda = 0$, with $\lambda > 0$, are $\sqrt[4]{\lambda}$, $i\sqrt[4]{\lambda}$, $-\sqrt[4]{\lambda}$, and $-i\sqrt[4]{\lambda}$. To simplify notation, you may want to use $\alpha := \sqrt[4]{\lambda}$.

12. Apply the method of separation of variables to find the general solution to the PDE

$$u_t + u_{xxxx} = 0,$$

with $0 < x < L$, subject to the boundary conditions $u(0,t) = u(L,t) = 0$ and $u_{xx}(0,t) = u_{xx}(L,t) = 0$. (You are not asked to take into account any initial conditions).

Hint: After separating variables, use the solution to the previous problem.