Math 465/501 Spring 2018 Homework Set 5

1. Shifrin (Page 89) Exercise 5.

Proof: Since M is compact, oriented surface, by Gauss-Bonnet (Corollary 1.11 Page 88 shifrin) $\iint_M K\,dA = 2\pi\chi(M)$.

Since M is not of the topological type of a sphere, so $\chi(M) \neq 2$, then $\chi(M) = 0, -2, \cdots, -2n, \cdots$ $n \in \mathbb{N}$.

Then $\iint_M K\,dA \le 0$.

Since M is compact, it has an ellptic point p and the curvature K_p at p is positive. Then since $\iint_M K\,dA \le 0$, there has to be a point q on M where $K_q < 0$.

Since the curvature on M is a continuous function, we can apply the Intermediate Value Theorem, hence there exists a point $r \in M$ where $K_r = 0$.

$\square$

2 Shifrin (Page 90) Exercise 8

(a) $x(u,v) = (u\cos(v), u\sin(v), u^2)$

$X_u = (\cos(v), \sin(v), 2u)$

$X_v = (-u\sin(v), u\cos(v), 0)$

$X_u \times X_v = (-2u^2\cos(v), -2u^2\sin(v), u)$

$$n = \frac{X_u \times X_v}{\|X_u \times X_v\|} = \frac{1}{\sqrt{4u^2+1}}(-2u\cos v, -2u\sin v, 1)$$

For the boundary circle, let r be the radius. then

$\alpha(v) = (r\cos v, r\sin v, r^2)$. Reparametrize α by arclength,

we have $\alpha(s) = (r\cos\frac{s}{r}, r\sin\frac{s}{r}, r^2)$ and $k = \frac{1}{r}$.

Then $\alpha'(s) = (-\sin\frac{s}{r}, \cos\frac{s}{r}, 0)$ and $T = \frac{\alpha'(s)}{\|\alpha'(s)\|} = (-\sin\frac{s}{r}, \cos\frac{s}{r}, 0)$.

Then $T' = (-\frac{1}{r}\cos\frac{s}{r}, -\frac{1}{r}\sin\frac{s}{r}, 0)$

$$N = \frac{T'}{\|T'\|} = (-\cos\frac{s}{r}, -\sin\frac{s}{r}, 0)$$

Then $n \times T = \frac{1}{\sqrt{4u^2+1}}(-\cos v, -\sin v, -2u)$

$$k_g = kN \cdot (n \times T) = \frac{1}{r}(-\cos\frac{s}{r}, -\sin\frac{s}{r}, 0) \cdot \frac{1}{\sqrt{4u^2+1}}(-\cos v, -\sin v, -2u)$$

$$= \frac{1}{r\sqrt{4r^2+1}}(-\cos\frac{s}{r}, -\sin\frac{s}{r}, 0) \cdot (-\cos\frac{s}{r}, -\sin\frac{s}{r}, -2r)$$

$$= \frac{1}{r\sqrt{4r^2+1}}$$

Then $\int_{\partial M_r} k_g \, ds = \int_0^{2\pi r} \frac{1}{r\sqrt{4r^2+1}} ds = 2\pi r \cdot \frac{1}{r\sqrt{4r^2+1}} = \frac{2\pi}{\sqrt{4r^2+1}}$.

(b) Notice that M_r is topologically equivalent to a closed disk, it has the same Euler Characterist as a disk, hence $\chi(M_r) = 1$.

(c) By Gauss - Bonnet we have:

$$\int_{\partial M_r} k_g \, ds + \int_{M_r} K \, dA = 2\pi \chi(M_r) = 2\pi.$$

then $$\int_{M_r} K \, dA = 2\pi - \frac{2\pi}{\sqrt{4r^2+1}}$$

$$\lim_{r\to\infty} \int_{M_r} K \, dA = \lim_{r\to\infty} 2\pi - \frac{2\pi}{\sqrt{4r^2+1}} = 2\pi$$

(d) $X_u = (\cos(v), \sin(v), 2u)$

$X_v = (-u\sin(v), u\cos(v), 0)$

$E = X_u \cdot X_u = 4u^2+1$

$F = X_u \cdot X_v = 0.$

$G = X_v \cdot X_v = u^2$

$$I_p = \begin{bmatrix} E & F \\ F & G \end{bmatrix} = \begin{bmatrix} 4u^2+1 & 0 \\ 0 & u^2 \end{bmatrix}$$

$X_{uu} = (0, 0, 2)$

$X_{uv} = (-\sin(v), \cos(v), 0)$

$X_{vv} = (-u\cos v, -u\sin v, 0)$

$$n = \frac{1}{\sqrt{4u^2+1}} (-2u\cos v, -2u\sin v, 1)$$

$l = X_{uu} \cdot n = \frac{2}{\sqrt{4u^2+1}}$

$m = X_{uv} \cdot n = 0$

$n = X_{vv} \cdot n = \frac{2u^2}{\sqrt{4u^2+1}}$

$$II_p = \begin{bmatrix} l & m \\ m & n \end{bmatrix} = \begin{bmatrix} \frac{2}{\sqrt{4u^2+1}} & 0 \\ 0 & \frac{2u^2}{\sqrt{4u^2+1}} \end{bmatrix}$$

$$K = \det(\mathrm{I}_p^{-1}\,\mathrm{II}_p) = \frac{1}{\det(\mathrm{I}_p)}\cdot\det(\mathrm{II}_p) = \frac{1}{u^2(4u^2+1)}\cdot\frac{4u^2}{4u^2+1} = \frac{4}{(4u^2+1)^2}$$

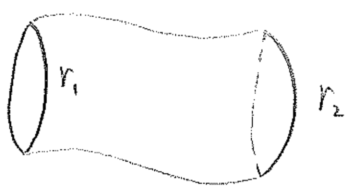
$$\iint_M K\,dA = \int_0^r\int_0^{2\pi} K\cdot\|X_u\times X_v\|\,du\,dv$$

$$= \int_0^{2\pi}\int_0^r \frac{4}{(4u^2+1)^2}\cdot u\sqrt{4u^2+1}\,du\,dv$$

$$= \int_0^{2\pi}\int_0^r \frac{4u}{(4u^2+1)^{3/2}}\,du\,dv$$

$$= 2\pi - \frac{2\pi}{\sqrt{4r^2+1}}$$

3. Proof: Suppose r_1 and r_2 are geodesics that do not intersect. They form a cylindrical surface with r_1, r_2 as boundaries.



Call this surface M. Then by Gauss-Bonet, we have:

$$\iint_M K\,dA + \int_{\partial M} k_g\,ds = 2\pi\chi(M).$$

Notice that M has the same Euler Characteristic as a cylinder, hence $\chi(M) = 0$.

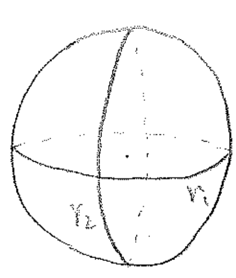
Since both r_1, r_2 are geodesics, then $\int_{\partial M} k_g\,ds = 0$.

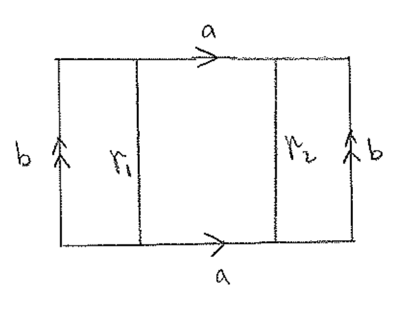
Then we have $\iint_M K\,dA = 0$.

In order for the integral above to be 0, we have to have $K = 0$.

Hence r_1, r_2 are boundaries of a flat region.

Example: Intersect: Sphere Not Intersect: Torus with flat metric





4. We are given that $K=-1$. By Gauss-Bonet Theorem, we have:

$\iint_{\Sigma_g} K\, dA = 2\pi \chi(\Sigma_g)$. Since $\chi(\Sigma_g) = 2-2g$, then we have

$$\iint_{\Sigma_g} -1\, dA = 2\pi(2-2g)$$

$$-A = 2\pi(2-2g)$$

$$A = 4\pi(g-1)$$