Math 465/501 Spring 2018 Homework Set #4

1. Shifrin (P76) Exercise 9

Claim: The only curves on a sphere that have constant geodesic curvature are the circles.

Proof: For a curve α on the sphere, suppose it has constant geodesic curvature. Then since $k^2 = k_g^2 + k_n^2$ (Exercise 2 page 75) we see that α has constant curvature. Then we apply Meusnier's Formula, (Proposition 2.5 Page 51), we get that the angle ϕ between N and $n = \alpha$ is constant, i.e. $\alpha \cdot N = \cos\phi = c$, where c is an arbitrary constant. Taking the derivative on both sides, we have: $\tau(\alpha \cdot B) = 0$.

Case 1: $\tau = 0$, thus α is a planer curve. A planer curve on a sphere is a circle.

Case 2: $\alpha \cdot B = 0$. Then $\alpha = \pm N$. So we have:

$\tau = N' \cdot B = \pm\alpha' \cdot B = \pm T \cdot B = 0$. This curve is a great circle on the sphere, hence a circle.

$\square$.

2 Shifrin (Page 76) Exercise 14

Proof: Let $\alpha(t) = X(u(t), v_0)$ $a \le t \le b$ be the u-curve from $X(u_0, v_0)$ to $X(u_1, v_0)$. We have the length of α as:

$$\text{length}(\alpha) = \int_a^b \sqrt{E u'(t)^2 + G v'(t)^2}\, dt = \int_a^b \sqrt{E u'(t)^2}\, dt.$$

To show that length(α) is independent of v, it suffices to show that E is independent of v.

We know that the curve we are looking at is a geodesic,

we also have $E = X_u \cdot X_u \Rightarrow E_v = 2 X_{uv} \cdot X_u$.

Using the formula on page 71 of the textbook, we have:

$$\begin{cases} u''(t) + \Gamma^u_{uu} u'(t)^2 = 0 \\ \Gamma^v_{uu} u'(t)^2 = 0 \end{cases} \Longrightarrow \Gamma^v_{uu} = 0 \Longrightarrow X_{uu} \cdot X_v = 0$$

Since $F = X_u \cdot X_v = 0$, we have $F_u = X_{uu} \cdot X_v + X_u \cdot X_{vu} = 0$.

then $X_u \cdot X_{uv} = 0$. Plug this into E_v, we have $E_v = 0$.

Thus $E = E(u)$, independent of v.

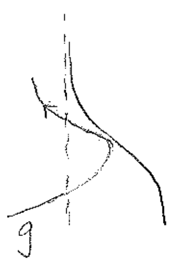
Therefore length(α) is independent of v. $\square$

3. Shifrin (page 77) : Exercise 18

(a) By proposition 4.4 (Clairaut's relation), we can see that the meridians are all geodesics. For each meridian, using notation in proposition 4.4, we have that $\phi = \frac{\pi}{2}$, hence $r\cos\phi = 0$.

The parallel with maximal radius r_0 is also a geodesic, as it satisfies $r_0 \cos\phi = r_0$ for $\phi = 0$.

For other geodesics, we must have $r\cos\phi = c$ where $c < r_0$. We claim that all these geodesics are bounded. To show our claim, consider the upper half of the surface:



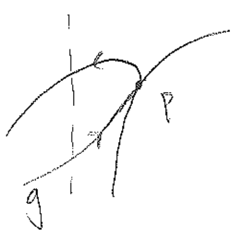
Let g be a geodesic that is unbounded. We have $r\cos\phi = c$. where c is constant. Notice that as g gets further from the central circle, $r \to 0$. In order for $r\cos\phi$ to remain constant, we need $\cos\phi \to \infty$ as $r \to 0$, This is impossible. Thus such geodesic does not exist.

We obtain same conclusion on lower half of surface by symmetry. Then we conclude that: All geodesics that are not meridians are bounded, the meridians are unbounded geodesics.

(b) By the same argument as in part (a), we have that the meridians and the central circle with smallest radius are geodesics

Claim that all other geodesics are unbounded.

To prove our claim: Consider the upper half of the surface. suppose g is a geodesic that is bounded.



Then at point P, $\phi = 0$. Since $r\cos\phi = c$ where c is a constant, and point P is a local max for $\cos\phi$ ($\cos\phi$ becomes negative passing P), then P has to be a local min for r. Observing the picture, P is not a local min, we have a contradiction. Thus g cannot be bounded.

We obtain the same conclusion on the lower half of the surface by symmetry. Then we conclude that: The only bounded geodesic is the central circle, all other geodesics are unbounded.

4. Shifrin (page 77) Exercise 21.

Proof: Let e_1^0, e_2^0 be a fixed orthonormal basis of T_pM. Since parallel translation on M is independent of paths, we can define vector fields e_1 and e_2 by parallel translating the e_1^0 and e_2^0. Then we can choose coordinates so that the u-curves are always tangent to e_1 and the v-curves are always tangent to e_2. Let M be parametrized by $X(u,v)$, then by construction $X_u \cdot X_v = F = 0$

$$\nabla_{e_1} e_1 = \nabla_{X_u} X_u = 0 = \Gamma_{uu}^{u} X_u + \Gamma_{uu}^{v} X_v \Rightarrow \Gamma_{uu}^{u} = \Gamma_{uu}^{v} = 0 \Rightarrow E_u = 0.$$

$$\nabla_{e_2} e_2 = \nabla_{X_v} X_v = 0 = \Gamma_{vv}^{u} X_u + \Gamma_{vv}^{v} X_v \Rightarrow \Gamma_{vv}^{u} = \Gamma_{vv}^{v} = 0 \Rightarrow G_v = 0.$$

Thus the u-curves are geodesics and the v-curves are also geodesics.

Then using the statement in Exercise 13 (b) (suppose $F = 0$ and the u-, v-curves are geodesics, the surface is flat), we conclude that M is flat.