Math 465/501 Spring 2018 Homework set 3

1. Shifrin (p64) Exercise 6.

Proof: Suppose M is a compact minimal surface in $\mathbb{R}^3$.

Consider the following function defined on M: $f: M \to \mathbb{R}$ defined as: $f(p) = |p|$, where $|\cdot|$ is the Euclidean Norm.

We know that the norm is continuous, i.e. f is a continuous function defined on M.

Since M is compact, f obtains a maximum at point $p_0 \in M$. This p_0 is a maximum in all directions on M, hence it is an elliptic point on M. Thus at p_0, the Gaussian Curvature $K_{p_0} > 0$.

Since M is minimal, we know that the Mean Curvature $H = 0$, in particular at point p_0. $H_{p_0} = 0$. This implies that at p_0, the principal curvatures satisfies the following relation:

$k_{1 p_0} + k_{2 p_0} = 0 \Rightarrow k_{1 p_0} = -k_{2 p_0}$.

But notice that $K_{p_0} = (k_{1 p_0})(k_{2 p_0}) = (-k_{2 p_0}) \cdot (k_{2 p_0}) = -k_{2 p_0}^2 \leq 0$.

We have a contradiction.

Therefore, there is no compact minimal surface $M \subset \mathbb{R}^3$.

$\square$

2. Shifrin (P65) Exercise 12

Proof: For surface $X(u,v) = (u\cos v, u\sin v, v)$:

$X_u = (\cos v, \sin v, 0)$

$X_v = (-u\sin v, u\cos v, 1)$

$E_X = X_u \cdot X_u = \cos^2 v + \sin^2 v = 1$.

$F_X = X_u \cdot X_v = -u\cos v\sin v + \cancel{u\cos v\sin v} + 0 = 0$.

$G_X = X_v \cdot X_v = u^2\sin^2 v + u^2\cos^2 v + 1 = u^2+1$.

$E_{Xv} = 0$ $\quad G_{Xu} = 2u$. $\quad E_X G_X = u^2+1$.

From the textbook Page 60 we have:

$$K_X = -\frac{1}{\sqrt{E_X G_X}}\left(\left(\frac{E_{Xv}}{\sqrt{E_X G_X}}\right)_v + \left(\frac{G_{Xu}}{\sqrt{E_X G_X}}\right)_u\right).$$

$$= -\frac{1}{\sqrt{u^2+1}}\left(0 + \left(\frac{2u}{\sqrt{u^2+1}}\right)_u\right)$$

$$= -\frac{1}{\sqrt{u^2+1}}\left(\frac{2}{(u^2+1)^{3/2}}\right)$$

$$= -\frac{2}{(u^2+1)^2}.$$

For surface $Y(u,v) = (u\cos v, u\sin v, \ln u)$.

$Y_u = (\cos v, \sin v, \frac{1}{u})$.

$Y_v = (-u\sin v, u\cos v, 0)$.

$E_Y = Y_u \cdot Y_u = \cos^2 v + \sin^2 v + \frac{1}{u^2} = \frac{1}{u^2} + 1$.

$F_Y = Y_u \cdot Y_v = -u\sin v\cos v + u\sin v\cos v + 0 = 0$

$G_Y = Y_v \cdot Y_v = u^2\sin^2 v + u^2\cos^2 v = u^2.$

$E_{Yv} = 0 \qquad G_{Yu} = 2u \qquad E_Y G_Y = 1+u^2$

Similarly we have:

$$K_Y = -\frac{1}{\sqrt{E_Y G_Y}}\left(\left(\frac{E_{Yv}}{\sqrt{E_Y G_Y}}\right)_v + \left(\frac{G_{Yu}}{\sqrt{E_Y G_Y}}\right)_u\right)$$

$$= -\frac{1}{\sqrt{1+u^2}}\left(0 + \left(\frac{2u}{\sqrt{1+u^2}}\right)_u\right)$$

$$= -\frac{1}{\sqrt{1+u^2}}\left(\frac{2}{(u^2+1)^{3/2}}\right)$$

$$= -\frac{2}{(1+u^2)^2}$$

Observing the computation above, we see that $K_X = K_Y$

but $I_X \neq I_Y$ since $E_X \neq E_Y$ and $G_X \neq G_Y$.

3 Shifrin (P66) Exercise 18

(a) Catenoid: $X(u,v) = (-\cosh u \sin v, \cosh u \cos v, u)$

$X_u = (-\sinh u \sin v, \sinh u \cos v, 1)$

$X_v = (-\cosh u \cos v, -\cosh u \sin v, 0)$

First fundamental form coefficients:

$E = X_u \cdot X_u = \sinh^2 u \sin^2 v + \sinh^2 u \cos^2 v + 1 = \sinh^2 u + 1 = \cosh^2 u.$

$F = X_u \cdot X_v = \sinh u \cosh u \sin v \cos v - \sinh u \cosh u \cos v \sin v = 0$

$G = X_v \cdot X_v = \cosh^2 u \cos^2 v + \cosh^2 u \sin^2 v = \cosh^2 u.$

$X_u \times X_v = (\cosh u \sin v, -\cosh u \cos v, \cosh u \sinh u)$

$\|X_u \times X_v\| = \sqrt{\cosh^2 u \sin^2 v + \cosh^2 u \cos^2 v + \cosh^2 u \sinh^2 u}$

$= \sqrt{\cosh^2 u + \cosh^2 u \sinh^2 u} = \sqrt{\cosh^2 u (1 + \sinh^2 u)}$

$= \sqrt{\cosh^4 u} = \cosh^2 u.$

$$N = \frac{X_u \times X_v}{\|X_u \times X_v\|} = \left(\frac{\sin v}{\cosh u}, \frac{-\cos v}{\cosh u}, \frac{\sinh u}{\cosh u} \right).$$

$X_{uu} = (-\cosh u \sin v, \cosh u \cos v, 0)$

$X_{uv} = (-\sinh u \cos v, -\sinh u \sin v, 0)$

$X_{vv} = (\cosh u \sin v, -\cosh u \cos v, 0)$

Second fundamental form coefficients:

$e = N \cdot X_{uu} = -\sin^2 v - \cos^2 v = -1$

$$f = N \cdot X_{uv} = -\frac{\sinh u \cos v \sin v}{\cosh u} + \frac{\sinh u \cos v \sin v}{\cosh u} = 0.$$

$g = N \cdot X_{vv} = \sin^2 v + \cos^2 v = 1$

The shape operator : $S_p = \begin{pmatrix} E & F \\ F & G \end{pmatrix}^{-1} \begin{pmatrix} e & f \\ f & g \end{pmatrix}$

Mean Curvature $= \frac{1}{2} tr(S_p) = \dfrac{eG - 2fF + Eg}{2(EG - F^2)}$

$$= \frac{-\cosh^2 u - 0 + \cosh^2 u}{2(\cosh^4 u - 0)} = 0.$$

Thus the surface is minimal.

<u>Helicoid</u> : $Y(u,v) = (u\cos v,\ u\sin v,\ v)$

$Y_u = (\cos v,\ \sin v,\ 0)$.

$Y_v = (-u\sin v,\ u\cos v,\ 1)$.

First fundamental form coefficients :

$E = Y_u \cdot Y_u = \cos^2 v + \sin^2 v = 1$

$F = Y_u \cdot Y_v = -u\sin v\cos v + u\sin v\cos v = 0$

$G = Y_v \cdot Y_v = u^2\sin^2 v + u^2\cos^2 v + 1 = u^2 + 1$.

$Y_u \times Y_v = (\sin v,\ -\cos v,\ u)$

$\|Y_u \times Y_v\| = \sqrt{\sin^2 v + \cos^2 v + u^2} = \sqrt{1 + u^2}$

$N = \dfrac{Y_u \times Y_v}{\|Y_u \times Y_v\|} = \left(\dfrac{\sin v}{\sqrt{1+u^2}},\ \dfrac{-\cos v}{\sqrt{1+u^2}},\ \dfrac{u}{\sqrt{1+u^2}} \right)$.

$Y_{uu} = (0,\ 0,\ 0)$ $\quad Y_{uv} = (-\sin v,\ \cos v,\ 0)$

$Y_{vv} = (-u\cos v,\ -u\sin v,\ 0)$

Coefficients of second fundamental form:

$$e = N \cdot Y_{uu} = 0.$$

$$f = N \cdot Y_{uv} = \frac{-\sin^2 v}{\sqrt{1+u^2}} - \frac{\cos^2 v}{\sqrt{1+u^2}} = -\frac{1}{\sqrt{1+u^2}}$$

$$g = N \cdot Y_{vv} = \frac{-u\sin v\cos v}{\sqrt{1+u^2}} + \frac{u\sin v\cos v}{\sqrt{1+u^2}} = 0.$$

The shape operator : $S_p = \begin{pmatrix} E & F \\ F & G \end{pmatrix}^{-1} \begin{pmatrix} e & f \\ f & g \end{pmatrix}$

$$\text{Mean Curvature} = \frac{1}{2}\operatorname{tr}(S_p) = \frac{eG - 2fF + Eg}{2(EG - F^2)}$$

$$= \frac{0 - 0 + 0}{2(u^2+1-0)} = 0.$$

Thus the surface is minimal.

(c) Rewrite the two surfaces in the following way:

Catenoid : $X(u,v) = (-\cosh u \sin v, \cosh u \cos v, u)$

Helicoid : $Y(w,z) = (w\cos z, w \sin z, z)$.

We do this to avoid confusion in notation.

Let $w = \sinh u$, $z = v$, i.e. $\psi(u,v) = (\sinh u, v)$.

First we show that ψ is a diffeomorphism. It is clearly bijective in each coordinate and $d\psi(u,v) = \begin{pmatrix} \cosh u & 0 \\ 0 & 1 \end{pmatrix}$ whose determinant is always positive, hence ψ is a diffeomorphism.

We want a regular parametrization $Y(u,v)$ of the helicoid, s.t $X = Y \circ \psi$. Since X is a regular parametrization, $dX(u,v)$ is an invertible 2×2 matrix, $d\psi(u,v)$ is also an invertible 2×2 matrix. By chain rule : $dX(u,v) = dY(\psi(u,v)) \cdot d\psi(u,v)$. Hence $dY(\psi(u,v))$ is also an invertible 2×2 matrix. Thus Y is also a regular parametrization.

Now we have a new parametrization of the helicoid:

$$Y(u,v) = (\sinh u \cos v, \sinh u \sin v, v)$$

$$Y_u = (\cosh u \cos v, \cosh u \sin v, 0)$$

$$Y_v = (-\sinh u \sin v, \sinh u \cos v, 1)$$

First fundamental form coefficients:

$$E = Y_u \cdot Y_u = \cosh^2 u \cos^2 v + \cosh^2 u \sin^2 v = \cosh^2 u$$

$$F = Y_u \cdot Y_v = -\sinh u \cosh u \sin v \cos v + \sinh u \cosh u \sin v \cos v = 0$$

$$G = Y_v \cdot Y_v = \sinh^2 u \sin^2 v + \sinh^2 u \cos^2 v + 1 = \sinh^2 u + 1 = \cosh^2 u.$$

Now we see that the catenoid and helicoid have the same first fundamental form, the two surfaces are locally isometric.

(d) Proof: Suppose they are, then the map φ that maps $X(u,v)$ to $Y(u,v)$ is bijective. But notice that

$X(u, v+2\pi) = X(u, v)$ hence $\varphi(X(u, v+2\pi)) = \varphi(X(u,v))$

so φ is not injective. We have a contradiction.

Therefore, the catenoid and helicoid are not globally isometric. $\square$

4. Shifrin (P 65) 19 (b)

(Reference Ex 9 on Page 52 of textbook)

Let S be a surface of revolution in $\mathbb{R}^3$ parametrized by $X(u,v) = (f(u)\cos v, f(u)\sin v, g(u))$, from class we have that $K = -\frac{f''}{f}$ since $K=1$ we have an ODE: $f''+f=0$.

By theory of 2^{nd} order ODE, we have the solution of the ODE above to be $f(u) = A\cos u + B\sin u$. for some $A, B \in \mathbb{R}$

$f(0) = A$ and $f'(0) = B$.

Since we have the parametrization by arclength, we have

$f'(u)^2 + g'(u)^2 = 1 \Rightarrow g'(u) = \sqrt{1 - f'(u)^2} = \sqrt{1-(-A\sin u + B\cos u)^2}$

Note: We need $|-A\sin u + B\cos u| < 1$.

Then $g(u) = \int_0^u \sqrt{1-f'(t)^2}\, dt$

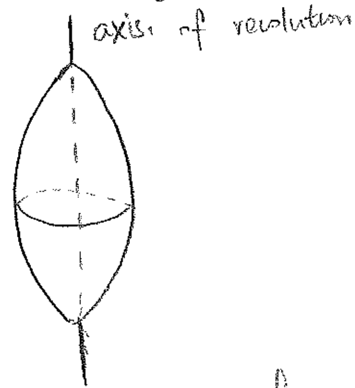
Hence $X(u,v) = \left((A\cos u + B\sin u)\cos v, (A\cos u + B\sin u)\sin v, \int_0^u \sqrt{1-f'(t)^2}\, dt\right)$

This shows that surface of revolution with $K=1$ is determind by A and $B \in \mathbb{R}$.

Notice that when the surface is closed $A=0$, when the surface is smooth when $B=1$.

If the surface is both smooth and closed, we have a sphere.

If $A/B > 1$, we get a resulting surface that is not smooth, something in this shape:



If $A/B < 1$, we get a resulting surface that is not closed, something in this shape: