Homework Set 1

DUE: FEB 1, 2017 (IN CLASS)

1. Shifrin (page 8): Exercise 4

2. Shifrin (page 10): Exercise 13

3. Shifrin (page 18): Exercise 3 (c)

4. Shifrin (page 18): Exercise 4

5. Shifrin (page 18): Exercise 7

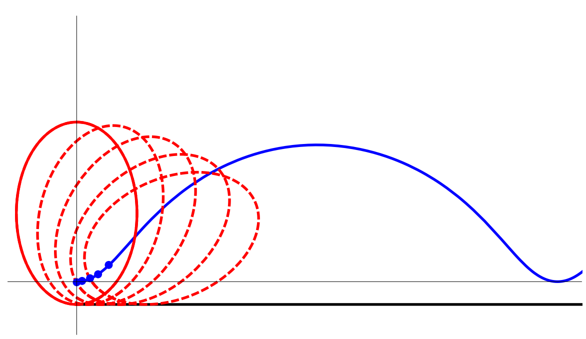
6. OPTIMAL REGULARITY FOR PARAMETRIZATION BY ARC LENGTH
   Show that if $\alpha\colon [t_0,t_1]\to\mathbb{R}^n$ is Lipschitz (i.e., each coordinate is a Lipschitz function), then $\alpha(t)$ admits a parametrization by arc length, that is, there exists an increasing function $S\colon [s_0,s_1]\to[t_0,t_1]$ such that the curve $\tilde\alpha\colon [s_0,s_1]\to\mathbb{R}^n$, $\tilde\alpha(s):=(\alpha\circ S)(s)$, satisfies $\|\tilde\alpha(s)-\tilde\alpha(s')\|=|s-s'|$ for all $s,s'\in[s_0,s_1]$.

   Remark: It can be shown that the least regularity needed for α to be *rectifiable*, i.e., for its length to be well-defined and finite, is that its coordinates are Lipschitz functions.

7. CHALLENGE PROBLEM 1 (OPTIONAL)
   The *roulette* γ_P of a point P along a curve α is the curve traced by the point P when α is rolled, without slipping, along a straight line. Let α be the ellipse $\frac{x^2}{a^2}+\frac{(y+c)^2}{b^2}=1$, where $0<a<b$ and $c=\sqrt{b^2-a^2}$, so that $P=(0,0)$ is one of the focal points of α. Show that the *roulette* γ_P in this case can be parametrized as:

$$\gamma_P(t)=\left(\int_0^t\sqrt{b^2-c^2\cos^2(z)}\,\mathrm{d}z-\frac{c\sin(t)(b-c\cos(t))}{\sqrt{b^2-c^2\cos^2(t)}},\frac{a(b-c\cos(t))}{\sqrt{b^2-c^2\cos^2(t)}}-b+c\right)$$

   Note: this curve γ_P is called an *undulary* or *elliptic catenary.* An example with $a=\frac{1}{2}$ and $b=1$ is plotted below, with α in red and γ_P in blue.



8. Challenge problem 2 (Optional)
   Write some code in your favorite mathematical software (e.g., Mathematica) that generates a plot like the above, and print out code and example plots.
   Note: you will need to use elliptic integrals to evaluate the x-coordinate of γ_P.