HW9 Solutions

1. $G(r,\theta) = (r\cos\theta,\ r\sin\theta)$

$$J = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix} = r\cos^2\theta + r\sin^2\theta = \boxed{r}$$

b. $G(r,\theta,z) = (r\cos\theta,\ r\sin\theta,\ z)$

$$J = \begin{vmatrix} \cos\theta & -r\sin\theta & 0 \\ \sin\theta & r\cos\theta & 0 \\ 0 & 0 & 1 \end{vmatrix} = r\cos^2\theta + r\sin^2\theta = \boxed{r}$$

c. $G(r,\theta,\varphi) = (r\cos\theta\sin\varphi,\ r\sin\theta\sin\varphi,\ r\cos\varphi)$

$$J = \begin{vmatrix} \cos\theta\sin\varphi & -r\sin\theta\sin\varphi & r\cos\theta\cos\varphi \\ \sin\theta\sin\varphi & r\cos\theta\sin\varphi & r\cos\varphi\sin\theta \\ \cos\varphi & 0 & -r\sin\varphi \end{vmatrix}$$

$$= \cos\theta\sin\varphi\,[r\cos\theta\sin\varphi(-r\sin\varphi)] - (-r\sin\theta\sin\varphi)\,[\sin\theta\sin\varphi(-r\sin\varphi) - \cos\varphi(r\sin\theta\cos\varphi)] + r\cos\theta\cos\varphi\,[-\cos\varphi(r\cos\theta\sin\varphi)] = -r^2\sin\varphi$$

$$\Rightarrow |-r^2\sin\varphi| = \boxed{r^2\sin\varphi}$$

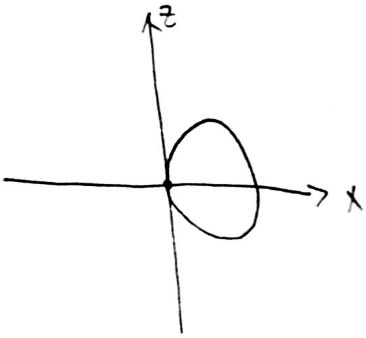
d. $G(u,v,w) = (au+1,\ bv-2,\ cw+3)$

$$J = \begin{vmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{vmatrix} = \boxed{|abc|}$$

e. $G(u,v) = \left(\frac{v-u}{2},\ \frac{u+v}{2}\right)$

$$J = \begin{vmatrix} -\frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{vmatrix} = -\frac{1}{4} - \frac{1}{4} = -\frac{1}{2} \Rightarrow \boxed{\frac{1}{2}}$$

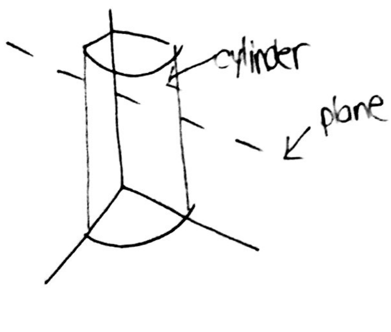
2. $r = \sin^{1/3}\varphi$



$$V = \int_0^{2\pi}\int_0^{\pi}\int_0^{\sin^{1/3}\varphi} r^2\sin\varphi\,dr\,d\varphi\,d\theta = \frac{1}{3}\int_0^{2\pi}\int_0^{\pi}\sin^2\varphi\,d\varphi\,d\theta = \frac{2\pi}{3}\int_0^{\pi}\sin^2\varphi\,d\varphi =$$

$$= \boxed{\frac{\pi^2}{3}}$$

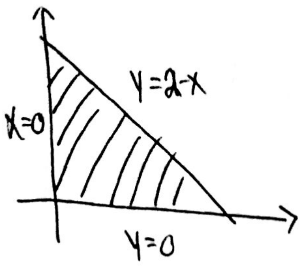
3.



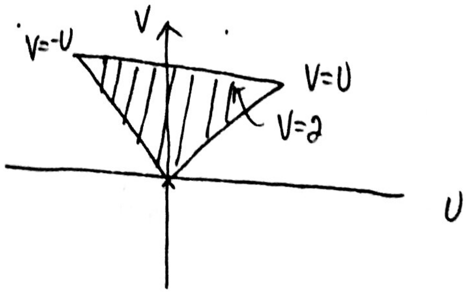
$$V = \int_0^{\pi/2}\int_0^{1}\int_0^{2-x-y\,=\,2-r\cos\theta-r\sin\theta} dz\,r\,dr\,d\theta = \int_0^{\pi/2}\int_0^{1}(2r - r^2\cos\theta - r^2\sin\theta)\,dr\,d\theta =$$

$$= \int_0^{\pi/2}\left(1 - \frac{1}{3}\cos\theta - \frac{1}{3}\sin\theta\right)d\theta = \boxed{\frac{\pi}{2} - \frac{2}{3}}$$

4.



$U = Y - X$
$V = Y + X$ $\Rightarrow$



$$J = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} \Rightarrow \begin{array}{l} x = \frac{V-U}{2} \\ y = \frac{U+V}{2} \end{array} \Rightarrow J = \begin{vmatrix} -\frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{vmatrix} = \left|-\frac{1}{2}\right| = \frac{1}{2}$$

$$\Rightarrow \frac{1}{2}\int_0^2 \int_{-v}^{v} e^{\frac{u}{v}}\, du\, dv = \frac{1}{2}\int_0^2 \left(ve - \frac{v}{e}\right)dv = \frac{1}{2}\left[2e - \frac{2}{e}\right] = \boxed{e - \frac{1}{e}}$$

5. $\int_r f(x,y,z)\,ds$: $\gamma(t) = (t, 1-t, 0)$, $t \in [0,1]$, $f(x,y,z) = 2x + 3y$ $\Rightarrow$ $f(t) = 2t + 3(1-t) = 3 - t$

$\gamma'(t) = (1, -1, 0)$ $\Rightarrow$ $|\gamma'(t)| = \sqrt{2}$

$$\int_0^1 \sqrt{2}\,(3-t)\,dt = \boxed{\frac{5}{2}\sqrt{2}}$$

b. $\gamma(t) = (\cos t, \sin t, 2t)$, $t \in [0, 2\pi]$, $f(x,y,z) = \sqrt{x^2+y^2}$ $\Rightarrow$ $f(t) = \sqrt{\cos^2 t + \sin^2 t} = 1$

$\gamma'(t) = (-\sin t, \cos t, 2)$

$|\gamma'(t)| = \sqrt{5}$ $\Rightarrow$ $\int_0^{2\pi} \sqrt{5}\,(1)\,dt = \boxed{2\pi\sqrt{5}}$

c. $\int \vec{F}\cdot dr$; $r(t) = (t^2, t, -t)$ $t \in [0,1]$, $\vec{F}(x,y,z) = (xy, -x, yz)$

$dr(t) = (2t, 1, -1)dt$, $\vec{F}(t) = (t^3, -t^2, -t^2)$

$$\int \vec{F}\cdot dr = \int_0^1 (2t, 1, -1)\cdot(t^3, -t^2, -t^2)\,dt = \int_0^1 2t^4 dt = \boxed{\frac{2}{5}}$$

d. $\int \vec{F}\cdot dr$; $r(t) = (\sin t, \cos t, t)$, $t \in [0, \pi]$, $\vec{F}(x,y,z) = (3z, y^2, 4x)$

$dr = (\cos t, -\sin t, 1)dt$ $\quad$ $\vec{F}(t) = (3t, \cos^2 t, 4\sin t)$

$$\int_0^{\pi} (3t\cos t - \sin t\cos^2 t + 4\sin t)\,dt = \frac{4}{3}$$