MATH 114 — Solution to Practice Problems for Midterm 2 — 3/21/2017

#1.

$\lim_{(x,y)\to(0,0)} \frac{xy}{x^2+y^2}$ does not exist: $x=0 \rightsquigarrow \lim_{y\to 0} \frac{0}{y^2} = 0.$

$x=y \longrightarrow \lim_{x\to 0} \frac{x^2}{2x^2} = \frac{1}{2}.$

$$\lim_{(x,y)\to(0,0)} \frac{x^2 \sin y}{x^2+y^2} = \lim_{(x,y)\to(0,0)} \underbrace{\frac{x^2}{x^2+y^2}}_{\text{bounded}} \cdot \underbrace{\sin y}_{\to 0} = 0.$$

$0 \le x^2 \le x^2+y^2 \implies 0 \le \frac{x^2}{x^2+y^2} \le 1$

$$\lim_{(x,y)\to(0,0)} \frac{x^5}{x^4+y^4} = \lim_{(x,y)\to(0,0)} \underbrace{\frac{x^4}{x^4+y^4}}_{\text{bounded}} \cdot \underbrace{x}_{\to 0} = 0$$

$\lim_{(x,y)\to(0,0)} \frac{x^3+y^3}{x^3-y^3}$ does not exist: $x=0 \rightsquigarrow \lim_{y\to 0} \frac{y^3}{-y^3} = -1$

$y=0 \longrightarrow \lim_{x\to 0} \frac{x^3}{x^3} = 1$

$$\lim_{(x,y)\to(0,0)} \frac{x-y}{x^3-y^3} = \lim_{(x,y)\to(0,0)} \frac{\cancel{x-y}}{\cancel{(x-y)}(x^2+xy+y^2)}$$ does not exist!

⚠ How about $(x,y)\to(1,1)$?

Note: Sign of x^2+xy+y^2 near $(0,0)$?

$f(0,0)=0$

$\nabla f(0,0)=(0,0)$

$\nabla^2 f(0,0) = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$

tr $=4$

det $= 3 > 0$ loc. min!

$\boxed{\lim = +\infty}$

#2

$f(x,y) = 3x^2y + y^3 - 3x^2 - 3y^2$

$$\nabla f(x,y) = (6xy - 6x,\ 3x^2+3y^2-6y) = (0,0) \iff \begin{cases} 6x(y-1) = 0 \\ 3(x^2+y^2) = 6y \end{cases}$$

$\iff x=0$ or $y=1$

$x=0 \Downarrow$: $3y^2-6y=0$, $y(y-2)=0$, $\Rightarrow y=0$ or $y=2$

$y=1$: $3x^2=3 \Rightarrow x=\pm 1$

Crit. pts.:

$(0,0)$, $(0,2)$,

$(-1,1)$, $(1,1)$

Hessian matrix is $\text{Hess } f(x,y) = \begin{pmatrix} 6y-6 & 6x \\ 6x & 6y-6 \end{pmatrix}$

At $(0,0)$: $\begin{pmatrix} -6 & 0 \\ 0 & -6 \end{pmatrix} \rightarrow$ loc. max

At $(0,2)$: $\begin{pmatrix} 6 & 0 \\ 0 & 6 \end{pmatrix} \rightarrow$ loc. min

At $(-1,1)$: $\begin{pmatrix} 0 & -6 \\ -6 & 0 \end{pmatrix} \rightarrow$ saddle

At $(1,1)$: $\begin{pmatrix} 0 & 6 \\ 6 & 0 \end{pmatrix} \rightarrow$ saddle.

#3

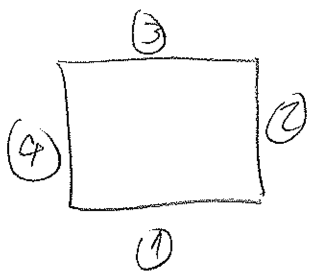
$f(x,y) = xy + x^2 - y^2$ on $[0,2]\times[0,2]$

Inside the region: $\nabla f(x,y) = (y+2x,\ x-2y) = (0,0) \Leftrightarrow \begin{cases} 2x + y = 0 \\ x - 2y = 0 \end{cases}$

$\Leftrightarrow x = y = 0.$

Only 1 crit. pt: $(0,0)$., $f(0,0) = 0$.

On the boundary:



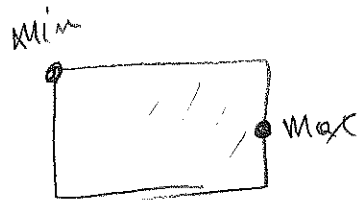
(1) $f(x,0) = x^2 \rightarrow$ min $= 0$ @ $x=0$, max $= 4$ @ $x=2$

(2) $f(2,y) = 2y + 4 - y^2 = -(1-y)^2 + 5$

$\rightarrow$ min $= 4$ @ $y = 0, 2$; max $= 5$ @ $y = 1$

(3) $f(x,2) = 2x + x^2 - 4 = (1+x)^2 - 5$

$\rightarrow$ min $= -4$ @ $x=0$; max $= 4$ @ $x=2$



(4) $f(0,y) = -y^2 \rightarrow$ min $= -4$ @ $y=2$; max $= 0$ @ $y=0$.

Thus, the absolute max is 5 ← attained @ $(x,y) = (2,1)$

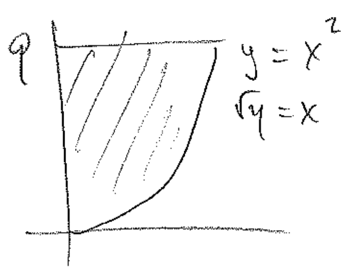
and the absolute min is -4. ← attained @ $(x,y) = (0,2)$

#4

a) $\int_0^1 \int_0^\pi x\cos(xy)\,dx\,dy = \int_0^\pi \int_0^1 x\cos(xy)\,dy\,dx = \int_0^\pi \sin(xy)\Big|_0^1 dx$

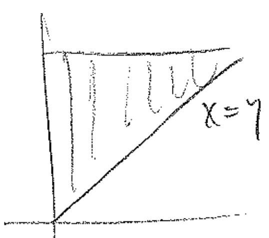
$= \int_0^\pi \sin x\,dx = \underline{\underline{2}}$

b) $\int_0^3 \int_{x^2}^9 \frac{x}{\sqrt{3x^2+y}}\,dy\,dx = \int_0^9 \int_0^{\sqrt{y}} \frac{x}{\sqrt{3x^2+y}}\,dx\,dy = \int_0^9 \frac{1}{3}\sqrt{3x^2+y}\Big|_0^{\sqrt{y}} dy$



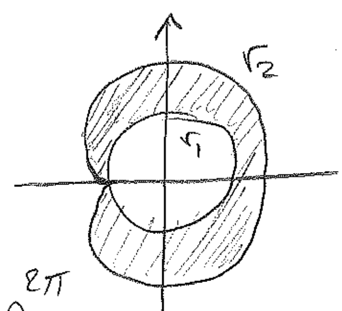
$= \frac{1}{3}\int_0^9 \underbrace{(2\sqrt{y} - \sqrt{y})}_{\sqrt{y}}\,dy = \frac{1}{3}\cdot\frac{2}{3}\,y^{3/2}\Big|_0^9 = \underline{\underline{6.}}$

c) $\int_0^\pi \int_x^\pi \frac{\cos y}{y}\,dy\,dx = \int_0^\pi \int_0^y \frac{\cos y}{y}\,dx\,dy = \int_0^\pi \frac{\cos y}{\cancel{y}}\cdot \cancel{y}\,dy = \underline{\underline{0}}$



d) $r_2(\theta) = 2 + \cos\theta$ (cardioid)

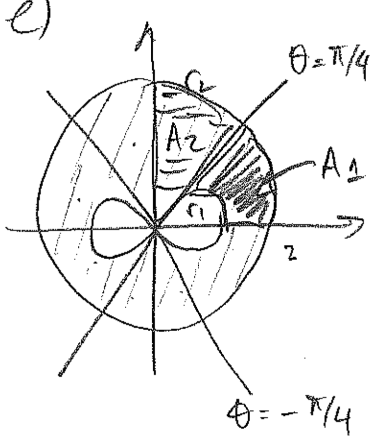
$r_1(\theta) = 1$ (circle)



$A = \int_0^{2\pi}\int_1^{2+\cos\theta} r\,dr\,d\theta = \int_0^{2\pi} \frac{r^2}{2}\Big|_1^{2+\cos\theta} d\theta = \int_0^{2\pi} \frac{1}{2}(2+\cos\theta)^2 - \frac{1}{2}\,d\theta$

$= \int_0^{2\pi} \frac{\cos^2\theta}{2} + \cos\theta + \frac{3}{2}\,d\theta = \frac{1}{2}\underbrace{\int_0^{2\pi}\cos^2\theta\,d\theta}_{\pi} + 3\pi = \underline{\underline{\frac{7\pi}{2}}}$

e)



$r_2(\theta) = 2$ (circle)

$r_1(\theta) = \cos 2\theta$ (lemniscate)

$A = 4A_1 + 4A_2$

$A_2 = \frac{4\pi}{8} = \frac{\pi}{2}, \quad A_1 = \int_0^{\pi/4}\int_{\sqrt{\cos 2\theta}}^2 r\,dr\,d\theta$

$$\therefore A_1 = \int_0^{\pi/4} \frac{r^2}{2}\Big|_{\sqrt{\cos 2\theta}}^{2} d\theta = \int_0^{\pi/4} 2 - \frac{\cos 2\theta}{2}\, d\theta = \frac{2\pi}{4} - \frac{1}{2}\int_0^{\pi/4} \cos 2\theta \, d\theta$$

$$= \frac{\pi}{2} - \left(\frac{\sin 2\theta}{4}\Big|_0^{\pi/4} \right) = \frac{\pi}{2} - \left(\frac{1}{4}\right).$$

thus $A = 4A_1 + 4A_2 = 4\left(\frac{\pi}{2} - \frac{1}{4}\right) + 4\left(\frac{\pi}{2}\right) = \underline{\underline{4\pi - 1}}$.

#5

$F(x,y,z) = x^2 + y^2 z^2$

Tangent plane to the levelset $F(x,y,z) = 8$ @ $(x,y,z) = (2,-1,2)$

$\nabla F(x,y,z) = (2x, 2yz^2, 2y^2 z) \Rightarrow \nabla F(2,-1,2) = (4,-8,4)$

Tangent plane is: $(4,-8,4)\cdot(x-2, y+1, z-2) = 0$

$$4(x-2) - 8(y+1) + 4(z-2) = 0$$

$$4x - 8y + 4z - 24 = 0$$

$$\boxed{x - 2y + z = 6}$$

Intersecting with z-axis $\Leftrightarrow$ $x = y = 0$ $\rightsquigarrow$ Intersection point is $(0,0,6)$.

#6 $z(x,y) = xe^{y-x}$ $\qquad \nabla z(x,y) = (e^{y-x} - xe^{y-x}, xe^{y-x})$

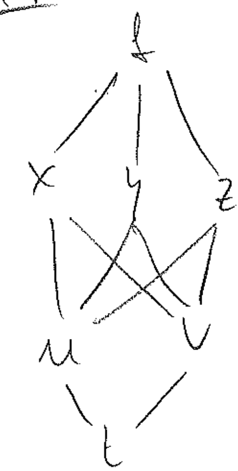
$\nabla z(1,1) = (1-1, 1) = (0,1)$

Linearization @ $(1,1)$ is;

$$L(x,y) = z(1,1) + \frac{\partial z}{\partial x}(1,1)(x-1) + \frac{\partial z}{\partial y}(1,1)(y-1) = 1 + (y-1) = y$$

$$z(1.1, 0.9) \cong L(1.1, 0.9) = \underline{\underline{0.9}}$$

#7

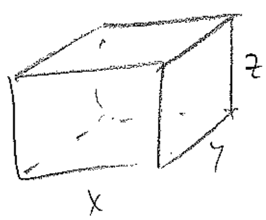


By the Chain Rule:

$$\frac{\partial f}{\partial t} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial u}\frac{du}{dt} + \frac{\partial f}{\partial x}\frac{\partial x}{\partial v}\frac{dv}{dt} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial u}\frac{du}{dt} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial v}\frac{dv}{dt}$$

$$+ \frac{\partial f}{\partial z}\frac{\partial z}{\partial u}\frac{du}{dt} + \frac{\partial f}{\partial z}\frac{\partial z}{\partial v}\frac{dv}{dt}.$$

#8



$V(x,y,z) = xyz = 16 \cdot 10^6 \, ft^3$ ← Volume (constant)

$C(x,y,z) = 55xz + 31xy + 2 \cdot 27yz + 27xz$ ← cost (target function)

$= 31xy + 54yz + 82xz$

$\nabla V = (yz, xz, xy)$

$\nabla C = (31y + 82z, 31x + 54z, 54y + 82x)$

$$\nabla C = \lambda \nabla V \iff \begin{cases} 31y + 82z = \lambda yz \\ 31x + 54z = \lambda xz \\ 54y + 82x = \lambda xy \end{cases}$$

solve for λ!

$$\lambda = \frac{31y + 82z}{yz} = \frac{31x + 54z}{xz} = \frac{54y + 82x}{xy}$$

Cross-multiply:

$$\begin{cases} \cancel{31xyz} + 82xz^2 = \cancel{31xyz} + 54yz^2 \\ 31xy^2 + \cancel{82xyz} = \cancel{82xyz} + 54y^2z \\ 31x^2y + \cancel{54xyz} = \cancel{54xyz} + 82x^2z \end{cases} \Rightarrow x = \frac{27}{41}y = \frac{54}{31}z$$

plug back to constant!

$$V = xyz = x\left(\frac{41}{27}x\right)\left(\frac{31}{54}x\right) = 16 \cdot 10^6 \Rightarrow x = \sqrt[3]{\frac{16 \cdot 10^6 \cdot 27 \cdot 54}{41 \cdot 31}}$$

$$x = \frac{1800 \cdot 2^{2/3}}{1271^{1/3}}$$

Thus, the least cost is choosing

$$x = \frac{1800 \cdot 2^{2/3}}{1271^{1/3}}, \quad y = \frac{200 \cdot 82^{2/3}}{3 \cdot 31^{1/3}}, \quad z = \frac{100 \cdot 62^{2/3}}{3 \cdot 41^{1/3}}$$

Approximating:

$$x \approx 263.78, \quad y \approx 400.56, \quad z \approx 151.43.$$

(Of course, these numbers may appear "ugly"... No need to simplify or estimate anything like that in the exam!)

#9

$A =$ Area $= xy = 5000$ (constraint)

$P =$ Perimeter $= 2x + y$ (target)

$$\nabla A = (y, x)$$

$$\nabla P = (2, 1)$$

$$\nabla A = \lambda \nabla P \iff \begin{cases} y = 2\lambda \\ x = \lambda \end{cases}$$

$$\iff \lambda = x = \frac{y}{2}$$

$$x \cdot \underbrace{(2x)}_{y} = 5000 \Rightarrow x^2 = 2500 \Rightarrow x = \pm 50 \quad (x = 50 \text{ b/c } x > 0)$$

$$x = 50, \quad y = 100.$$

Thus, least amount of fencing is $P(50, 100) = \underline{200 \text{ yards}}$.

#10

$B = 20x + 30y = 600$ budget (constraint)

$U = 10 x^{0.6} y^{0.4}$ utility (target)

$$\nabla B = (20, 30)$$

$$\nabla U = \left(6 x^{-0.4} y^{0.4}, 4 x^{0.6} y^{-0.6}\right)$$

$$\nabla U = \lambda \nabla B \iff \begin{cases} 6 x^{-0.4} y^{0.4} = 20\lambda \\ 4 x^{0.6} y^{-0.6} = 30\lambda \end{cases}$$

$$\lambda = \frac{3}{10} x^{-0.4} y^{0.4} = \frac{2}{15} x^{0.6} y^{-0.6}$$

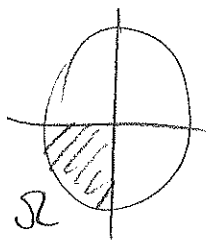
$$\Rightarrow \frac{45}{20} x^{-1} y = 1 \Rightarrow \frac{y}{x} = \frac{20}{45} = \frac{4}{9} \Rightarrow \underline{\underline{y = \frac{4x}{9}}}$$

$$20x + 30\left(\frac{4x}{9}\right) = 600 \Rightarrow 2x + \frac{4}{3}x = 60 \Rightarrow \frac{3x}{3} + \frac{2}{3}x = 30$$

$$\frac{5x}{3} = 30$$

$$\Rightarrow \underline{\underline{x = 18}}, \ \underline{\underline{y = 8}}$$

#11



Ω

$$\Omega = \{x^2+y^2 \leq 4 ; \ xy \leq 0\} = \left\{(r\cos\theta, r\sin\theta) ; \begin{array}{l} 0 \leq r \leq 2 \\ \pi \leq \theta \leq \frac{3\pi}{2} \end{array}\right\}.$$

$$V = \iint_{\Omega} (x^2+y^2)\, dx\,dy = \int_{\pi}^{3\pi/2} \int_0^2 r^2 \cdot r\,dr\,d\theta$$

$$= \frac{\pi}{2} \cdot \frac{r^4}{4}\Big|_0^2 = \underline{\underline{2\pi}}$$

#12

$$f(x,y) = \ln(x^2+y^2) \qquad \nabla f(x,y) = \left(\frac{2x}{x^2+y^2}, \frac{2y}{x^2+y^2}\right) \qquad \nabla f(1,1) = (1, 1)$$

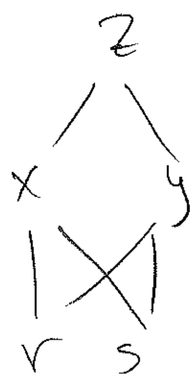
a) $$\frac{\partial f}{\partial \vec{v}}(1,1) = \nabla f(1,1) \cdot \vec{v} = (1,1) \cdot (1,-1) = 0.$$

b) $$\left|\frac{\partial f}{\partial \vec{v}}(1,1)\right| = |\nabla f(1,1) \cdot \vec{v}| = |\nabla f(1,1)| \cdot 1 \cdot \cos\theta \leq |\nabla f(1,1)| = \sqrt{2}$$

max achieved @ $\theta = 0$, i.e., $\vec{v} = \frac{\nabla f(1,1)}{|\nabla f(1,1)|}$,

Largest value of $\frac{\partial f}{\partial \vec{v}}(1,1)$ among vectors $\vec{v}$ with $|\vec{v}| = 1$ is $\underline{\underline{\sqrt{2}}} = |\nabla f(1,1)|$.

#13.



$$\frac{\partial z}{\partial r} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial r}$$

$$\frac{\partial z}{\partial r} = 2\cos(2x-y)\cdot 1 - \cos(2x-y)\cdot s$$

If $r=\pi$, $s=0$, we get $x=\pi$, $y=0$ so

$$\frac{\partial z}{\partial r} = 2\cos 2\pi - (\cos 2\pi)\cdot 0 = \underline{\underline{2}}.$$