Math 114, Spring 2017

Practice Problems for the Final Exam

1. The planes $3x + 2y + z = 6$ and $x + y = 2$ intersect in a line ℓ. Find the distance from the origin to ℓ.
   (Answer: $\frac{\sqrt{24}}{3}$)

2. Find the area of the triangle with vertices $A = (0, 0, 1)$, $B = (1, 2, 1)$ and $C = (0, 0, 0)$. What is the angle between the two edges AB and AC?
   (Answer: area is $\frac{\sqrt{5}}{2}$ and the angle is $\frac{\pi}{2}$)

3. A missile is launched from the top of a $15m$ high cliff. The missile reaches a maximum height of $20m$ and lands $60m$ away from its initial position. Find the missiles initial velocity.
   (Answer: Initial velocity is $\vec{v}_0 = (20, 10)$)

4. A particle has an acceleration $\vec{a}(t) = t\vec{i} + t^2\vec{j} + 2\vec{k}$. If its initial velocity is $\vec{v}_0 = (1, 3, 7)$ and it is initially at the origin, find its position function $\vec{r}(t)$.
   (Answer: $\vec{r}(t) = (1, 3, 7)t + (t^3/6, t^4/12, t^2)$)

5. Find the arc length of the curve $\vec{r}(t) = (t^2, \cos t + t \sin t, \sin t - t \cos t)$ for $0 \leq t \leq \sqrt{2}$.
   (Answer: $\sqrt{5}$)

6. Consider the helix $\vec{r}(t) = (3 \cos t, 3 \sin t, 4t)$, compute:
   a) $\vec{T}, \vec{N}$ and $\vec{B}$ at time $t = 0$;
   b) The curvature κ at time $t = 0$;
   c) a_T and a_N where $\vec{a} = a_T\vec{T} + a_N\vec{N}$ at time $t = 0$.

   (Answer: $\vec{T}(0) = (0, 3/5, 4/5)$, $\vec{N}(0) = (-1, 0, 0)$, $\vec{B}(0) = (0, -4/5, 3/5)$, $\kappa = 3/25$, $a_T = 0$ and $a_N = 3$)

7. Consider the function

$$f(x,y) = \begin{cases} (y+1)e^{-(x^2+y^2)} \sin(x^2+y^2) & \text{if} \quad y \geq 0, \\ e^{-(x^2+y^2)} \sin(x^2+y^2) & \text{if} \quad y < 0. \end{cases}$$

   a) Compute $\lim_{(x,y)\to(0,0)} f(x,y)$ if it exists.
   b) Is f continuous at $(0, 0)$?
   (Answer: 0; yes)

8. Consider the function

$$f(x,y) = \begin{cases} x + y\sin\left(\frac{1}{x}\right) & \text{if} \quad x \neq 0, \\ 0 & \text{if} \quad x = 0. \end{cases}$$

   a) Compute $\lim_{(x,y)\to(0,0)} f(x,y)$ if it exists.
   b) Is f continuous at $(0,0)$?
   (Answer: 0; yes)

9. Consider the function

$$f(x,y) = \begin{cases} \dfrac{xy^3}{x^4+y^4} & \text{if } (x,y) \neq (0,0) \\ 0 & \text{if } (x,y) = (0,0) \end{cases}$$

   a) Compute, if they exist, the partial derivatives f_x and f_y at $(0,0)$.
   b) Is the function $f(x,y)$ continuous at $(0,0)$?
   (Answer: $f_x(0,0) = f_y(0,0) = 0$; no)

10. Find the extrema of the function $f(x,y) = e^{-\frac{1}{3}x^3+x-y^2}$.
(Answer: saddle point at $(-1,0)$; local max at $(1,0)$)

11. Use the method of Lagrange multipliers to solve the following optimization problem. Find the isosceles triangle of largest area inscribed in the circle $x^2+y^2=1$ with the vertex between the two equal sides in the point $(0,1)$.
(Answer: the equilateral triangle, i.e., the triangle with vertices in $(0,1)$, $(\sqrt{3}/2,-1/2)$, $(-\sqrt{3}/2,-1/2)$)

12. Find the extrema of the function $f(x,y,z) = y-2z$ on the curve defined by the equations

$$2x - z = 2, \qquad x^2+y^2=1.$$

(Answer: extrema at $(-4/\sqrt{17}, 1/\sqrt{17}, -8/\sqrt{17}-2)$ and $(4/\sqrt{17}, -1/\sqrt{17}, 8/\sqrt{17}-2)$)

13. Compute the volume of the body delimited by the lower hemisphere $x^2+y^2+z^2=1$, $-1 \leq z \leq 0$, and the cone $z = 1-\sqrt{x^2+y^2}$, $0 \leq z \leq 1$.
(Answer: π)

14. Find the work done by the vector field $\vec{F} = (x^2, yz, y^2)$ on a particle moving along the path $\vec{r}(t) = 3t\vec{j} + 4t\vec{k}$ with $0 \leq t \leq 1$.
(Answer: $W = 24$.)

15. Compute
$$\int_C \sqrt{x+y}\,\mathrm{d}x$$
where C is the path that starts at $(0,0)$, then moves in a straight line to $(1,3)$, then moves in a straight line to $(0,3)$ and finally moves back to $(0,0)$ in a straight line.
(Answer: $2\sqrt{3}-4$.)

16. Use Green's Theorem to compute the following line integrals (all curves are positively oriented):

   a) $\displaystyle\int_\gamma \left(y+e^{\sqrt{x}}\right)\mathrm{d}x + \left(2x+\cos(y^2)\right)\mathrm{d}y$, where γ is the boundary of the region enclosed by $y=x^2$ and $x=y^2$.
   (Answer: 1/3)

   b) $\displaystyle\int_\gamma xy\,\mathrm{d}x + 2x^2\,\mathrm{d}y$, where γ consists of the line segment joining $(-2,0)$ to $(2,0)$ and the semicircle $x^2+y^2=4$, $y\geq 0$.
   (Answer: 0)

   c) $\displaystyle\int_\gamma 2xy\,\mathrm{d}x + x^2\,\mathrm{d}y$, where γ is the cardioid $r(\theta)=1+\cos\theta$.
   (Answer: 0)

   d) $\displaystyle\int_\gamma \left(xy+e^{x^2}\right)\mathrm{d}x + \left(x^2-\ln(1+y)\right)\mathrm{d}y$, where γ is the closed curve formed by the line segment joining $(0,0)$ to $(\pi,0)$ and $y=\sin x$.
   (Answer: π)

   e) $\displaystyle\int_\gamma \vec{F}\,\mathrm{d}\gamma$, where $\vec{F}(x,y)=(y^2-x^2y)\vec{\imath}+xy^2\vec{\jmath}$ and γ consists of the line segments joining the origin to $(2,0)$ and $(\sqrt{2},\sqrt{2})$ and the circular arc $x^2+y^2=4$ from $(2,0)$ to $(\sqrt{2},\sqrt{2})$.
   (Answer: $\pi+\frac{16}{3}(\frac{1}{\sqrt{2}}-1)$)

17. Use Green's Theorem to compute the area of $\Omega=\{(x,y)\in\mathbb{R}^2 : x^{2/3}+y^{2/3}\leq a^{2/3}\}$.
(Answer: $\frac{3\pi a^2}{8}$)

18. Let $\vec{F}\colon\mathbb{R}^2\to\mathbb{R}^2$ be a smooth radial vector field, that is, $\vec{F}(x,y)=f(r)\,\vec{r}$ where $\vec{r}=x\vec{\imath}+y\vec{\jmath}$, $r=\sqrt{x^2+y^2}$, and $f\colon\mathbb{R}\to\mathbb{R}$ is a smooth function. Show that $\vec{F}$ is conservative.

19. Determine whether the following vector fields $\vec{F}$ are conservative in the domain Ω. In the affirmative case, find a potential φ such that $\vec{F} = \nabla\varphi$.

   a) $\vec{F}(x,y) = (2xe^y + y, x^2e^y + x - 2y)$, $\Omega = \mathbb{R}^2$
   (Answer: $\vec{F}$ is conservative, $\varphi(x,y) = x^2e^y + xy - y^2$)

   b) $\vec{F}(x,y,z) = (2x^2 + 8xy^2, 3x^3y - 3xy, -4z^2y^2 - 2x^3z)$, $\Omega = \mathbb{R}^3$
   (Answer: $\vec{F}$ is not conservative)

   c) $\vec{F}(x,y,z) = (y^2\cos x + z^3, -4 + 2y\sin x, 3xz^2 + 2)$, $\Omega = \mathbb{R}^3$
   (Answer: $\vec{F}$ is conservative, $\varphi(x,y,z) = y^2\sin x + xz^3 - 4y + 2z$)

   d) $\vec{F}(x,y) = \left(\frac{-y}{x^2+y^2}, \frac{x}{x^2+y^2}\right)$, $\Omega = \mathbb{R}^2 \setminus \{(0,0)\}$
   (Answer: $\vec{F}$ is not conservative)

   e) $\vec{F}(x,y) = \left(\frac{-y}{x^2+y^2}, \frac{x}{x^2+y^2}\right)$, $\Omega = \mathbb{R}^2 \setminus \{(x,0) : x \leq 0\}$
   (Answer: $\vec{F}$ is conservative, $\varphi(x,y) = \arctan(\frac{y}{x})$)

   f) $\vec{F}(x,y) = \left(\frac{x}{x^2+y^2}, \frac{y}{x^2+y^2}\right)$, $\Omega = \mathbb{R}^2 \setminus \{(0,0)\}$
   (Answer: $\vec{F}$ is conservative, $\varphi(x,y) = \ln\sqrt{x^2+y^2}$)

20. Find a parametrization and use it to compute the area of the following surfaces Σ:

   a) Σ is the part of the sphere $x^2+y^2+z^2 = 4$ that lies inside the cone $z \geq \sqrt{x^2+y^2}$
   (Answer: $4\pi(2-\sqrt{2})$)

   b) Σ is the part of the plane $z = 2x + 3y$ that lies inside the cylinder $x^2 + y^2 = 16$
   (Answer: $16\pi\sqrt{14}$)

   c) Σ is the part of the cylinder $x^2+z^2 = a^2$ that lies inside the cylinder $x^2+y^2 = a^2$, where $a > 0$
   (Answer: $8a^2$)

   d) Σ is the part of the sphere $x^2+y^2+z^2 = a^2$ that lies inside the cylinder $x^2+y^2 = ax$, where $a > 0$
   (Answer: $2a^2(\pi - 2)$)

21. Compute the following surface integrals $\displaystyle\iint_\Sigma \vec{F}\,\mathrm{d}\Sigma$

   a) $\vec{F} = (x^2y, -3xy^2, 4y^3)$ and Σ is the part of the paraboloid $z = 9 - x^2 - y^2$, $z \geq 0$, oriented so that the unit normal vector at $(0,0,0)$ is $\vec{k}$
   (Answer: 0)

   b) $\vec{F} = (x, xy, xz)$ and Σ is the part of the plane $3x + 2y + z = 6$ inside the cylinder $x^2 + y^2 = 1$, oriented so that the unit normal vector is $\frac{1}{\sqrt{14}}(3,2,1)$
   (Answer: $-\frac{3\pi}{4}$)

c) $\vec{F} = (-y, x, 3z)$ and Σ is the hemisphere $z = \sqrt{16 - x^2 - y^2}$, oriented so that the unit normal at the point $(0, 0, 4)$ is $\vec{k}$
(Answer: 128π)

d) $\vec{F} = (-yz, 0, 0)$ and Σ is the part of the sphere $x^2+y^2+z^2 = 4$ outside the cylinder $x^2 + y^2 \leq 1$, oriented so that the unit normal at the point $(2, 0, 0)$ is $\vec{i}$
(Answer: 0)

e) $\vec{F} = (x, y, -2z)$ and Σ is the part of the cone $z = \sqrt{x^2 + y^2}$ bounded by the cylinder $x^2 + y^2 = 2x$, oriented so that its unit normal satisfies $\vec{n} \cdot \vec{k} < 0$
(Answer: $\frac{32}{3}$)

22. Use Stokes' Theorem to compute $\displaystyle\int_\gamma \vec{F}\, d\gamma$ in the following cases, where γ is always oriented so that its projection on the xy-plane is oriented counterclockwise.

a) $\vec{F}(x, y, z) = (xz, 2xy, 3xy)$ and γ is the boundary of the part of the plane $3x + y + z = 3$ contained in the first octant
(Answer: $\frac{7}{2}$)

b) $\vec{F}(x, y, z) = (z^2 + e^{x^2}, y^2 + \ln(1 + y^2), xy + \sin(z^3))$ and γ is the boundary of the triangle with vertices $(1, 0, 0)$, $(0, 1, 0)$ and $(0, 0, 2)$
(Answer: $\frac{4}{3}$)

c) $\vec{F}(x, y, z) = (x + \cos(x^3), y, x^2 + y^2 + z^{100})$ and γ is the boundary of the paraboloid $z = 1 - x^2 - y^2$ contained in the first octant
(Answer: 0)

d) $\vec{F}(x, y, z) = (y + z, 2x + (1 + y^2)^{20}, x + y + z)$ and γ is the intersection of the cylinder $x^2 + y^2 = 2y$ with the plane $z = y$
(Answer: π)

23. Use Gauss' Theorem to compute the following surface integrals $\displaystyle\iint_\Sigma \vec{F}\, d\Sigma$ in the following cases, where Σ is always oriented with outward pointing normal vector.

a) $\vec{F}(x, y, z) = (x^2z^3, 2xyz^3, xz^4)$ and Σ is the boundary surface of the parallelepiped with vertices $(\pm1, \pm2, \pm3)$
(Answer: 0)

b) $\vec{F}(x, y, z) = (yz \sin^3(x), y^2z \sin^2(x) \cos(x), 2yz^2 \sin^2(x) \cos(x))$ and Σ is the boundary surface of the parallelepiped with vertices $(\pm\pi, \pm1, \pm1)$
(Answer: 0)

c) $\vec{F}(x, y, z) = (x, y, z)$ and Σ is the sphere $x^2 + y^2 + z^2 = 4$
(Answer: 32π)

d) $\vec{F}(x, y, z) = (-y, x, z)$ and Σ is the sphere $x^2 + y^2 + z^2 = 1$
(Answer: $\frac{4\pi}{3}$)