4 Integration techniques

We are now out of Part I of the course, where everything goes back to number sense, and into a segment of the course that involves learning a skill. It's a high level skill, but you're good at that kind of thing or you wouldn't be here. So relax and enjoy some clean and satisfying computation. The material corresponds to Sections 5.5, 5.6 and 8.2 of the textbook. The first method ought to be review, and the second ought to be new, though due to your varied backgrounds some might find both to be review or both to be new.

Curricular note: in Math 104 they spend a whole lot of time on integration techniques, nearly half the course (rmember the slide I showed you of the Math 104 final?). These days you can get your computer (or even your Wolfram Alpha iPhone app) to do this for you so there isn't as great a need. But you need some familiarity in order to make sense of things, and learning the two most pervasive techniques strikes a reasonable balance.

4.1 Substitution

The most common way of doing a integral by substitution, and the only way for indefinite integrals, is as follows.

1. Change variables from x to u (hence the common name "u-substitution")
2. Keep track of the relation between dx and du
3. If you chose correctly you can now do the u-integral
4. When you're done, substitute back for x

The most common substitution is when you let $u = h(x)$ for some function h. Then $du = h'(x)\,dx$. Usually you don't do this kind of substitution unless there will be an $h'(x)\,dx$ term waiting which you can then turn into du. Also, you don't do this unless the rest of the occurences of x can also be turned into u. If h has an inverse function, you can do this by substituting $h^{-1}(u)$ for x everywhere. Now when you reach the fourth step, it's easier because you can just plug in $u = h(x)$ to get things back in terms of x.

Please read the examples in Section 5.5 – there are a ton. I will give just one.

EXAMPLE: Compute $\int \sin^n x \, \cos x \, dx$.

Solution: substitute $u = \sin x$ and $du = \cos x \, dx$. This turns the integral into $\int u^n \, du$ which is easily valuated as $u^{n+1}/(n+1)$. Now plug back in $u = \sin x$ and you get the answer

$$\frac{\sin^{n+1} x}{n+1} .$$

You might think to worry whether the substitution had the right domain and range, was one to one, etc., but you don't need to. When computing an indefinite integral you are computing an anti-derivative and the proof of correctness is whether the derivative is what you started with. You can easily check that the derivative of $\sin^{n+1} x/(n+1)$ is $\sin^n x \cos x$. There are a zillion examples of this in Section 5.5.

When evaluating a definite integral you can compute the indefinite integral as above and then evaluate. A second option is to change variables, including the limit of integration, and then never change back.

EXAMPLE: Compute $\int_1^2 \frac{x}{x^2+1} \, dx$.

If we let $u = x^2 + 1$ then $du = 2x \, dx$, so the integrand becomes $(1/2) \, du/u$. If x goes from 1 to 2 then u goes from 2 to 5, thus the integral becomes

$$\int_2^5 \frac{1}{2} \frac{du}{u} = \frac{1}{2}(\ln 5 - \ln 2) .$$

Of course you can get the same answer in the usual way: the indefinite integral is $(1/2) \ln u$; we substitute back and get $(1/2) \ln(x^2 + 1)$. Now we evaluate at 2 and 1 instead of 5 and 2, but the result is the same: $(1/2)(\ln 5 - \ln 2)$.

Some useful derivatives

A large part of exact integration is recognizing when something is a derivative of something familiar. Here is a list of functions whose derivatives you should stare at long enough to recognize if they come up. (Yes, you can put them on a cheatsheet when exam time comes.)

$$
\begin{aligned}
\frac{d}{dx}\tan x &= \sec^2 x \\
\frac{d}{dx}\sec x &= \sec x \tan x \\
\frac{d}{dx}\arcsin x &= \frac{1}{\sqrt{1-x^2}} \\
\frac{d}{dx}\arctan x &= \frac{1}{1+x^2} \\
\frac{d}{dx}\operatorname{arcsinh} x &= \frac{1}{\sqrt{1+x^2}} \\
\frac{d}{dx}\operatorname{arccosh} x &= \frac{1}{\sqrt{x^2-1}} \\
\frac{d}{dx}\operatorname{arctanh} x &= \frac{1}{1-x^2}
\end{aligned}
$$

4.2 Integration by parts

The integral by parts formula

$$\int u\,dv = uv - \int v\,du$$

is pretty well explained in Section 8.2 of the textbook. Here I will just mention a couple of the trickier instances of integration by parts.

Repeated integration by parts

As you will see, when one of the functions involved is e^x, and you take $dv = e^x\,dx$, then $v\,du$ will still have an e^x in it. In that case you can integrate by parts again. Will this ever stop? Well if the original u is a polynomial $p(x)$ then du will be $p'(x)\,dx$ so it will have degree one less, and if you repeat enough times you'll get to stop eventually.

Similarly, if $dv = \sin x \, dx$ or $\cos x \, dx$, then the v term will just cycle through sines and cosines and if it's multiplied by a polynomial, the degree will go down each time you integrate by parts and eventually you'll get an answer.

You could make an algorithm dealing will all integrals of the form $p(x) \, dx$. The book does exactly this and calls it **tabular integration**. If you want to learn to integrate $p(x)e^x$, $p(x) \sin x$ and $p(x) \cos x$ this way, go ahead. As far as I'm concerned, it is just as easy to do it out. Well if p has very high degree then probably you'll want to make up some kind of shortcut, but I don't insist that it be the book's version of tabular integration. I will give just one example because this is something the book handles well.

Example: Integrate $\int (x^3 + 3x) \cos x \, dx$. Taking $u = x^3 + 3x$ and $dv = \cos x \, dx$ gives

$$\int (x^3 + 3x) \cos x \, dx = (x^3 + 3x) \sin x - \int (3x^2 + 3) \sin x \, dx \, .$$

Setting aside the first term on the right-hand side, we work on the second, integreating by parts again. There will be fewer double negatives if we take the minus sign inside and attach it to the $\sin x \, dx$ term:

$$\int (3x^2 + 3)(-\sin x) \, dx = (3x^2 + 3) \cos x - \int 6x \cos x \, dx \, .$$

One last integration by parts shows that the last integral is

$$6x \sin x - \int 6 \sin x dx = 6x \sin x + 6 \cos x \, .$$

So the whole thing comes out to be

$$(x^3 + 3x) \sin x + (3x^2 + 3) \cos x - (6x \sin x + 6 \cos x) \, .$$

You can do all of this when there is a term like e^{3x} or $\cos(5x)$. Technically this is a substitution plus an integration by parts but when the substitution is just $5x$ for x, you can pretty much do it in your head. For example to integrate $\int xe^{5x} \, dx$ you can let $dv = e^{5x} \, dx$ and therefore $v = (1/5)e^{5x}$. The substitution is hidden in the correct evaluation of v from dv.

Back where you started but with a sign change

If you try to integrate $e^x \sin x$, you'll find you have a choice. You can make $u = e^x$ and $dv = \sin x\, dx$ or $u = \sin x$ and $dv = e^x\, dx$. Either way, if you do it twice, you're get back to where you started but with the opposite sign. That's good because you have something like

$$\int e^x \sin x\, dx = \text{otherstuff} - \int e^x \sin x\, dx\,.$$

Now you can move the last term to the right over to the left so it becomes twice the integral you want, and see that the integral you want is half of the other stuff on the right. Example 4 on page 464 of the textbook is a very clear description of this.

Last trick: you can always try $dv = dx$

When trying to integrate $\int f(x)\, dx$ it doesn't look like there's a u and a v but if you know the derivative of f you can always let $u = f$ and $dv = dx$ and get

$$\int f(x)\, dx = xf(x) - \int xf'(x)\, dx\,.$$

Whether this helps depends on whether the factor of x combines nicely with the f'. The easiest example that works out nicely is $\int \ln x\, dx$ which is Example 2 on page 463 of the textbook.