1 Functions

If we count AB calculus as a pre-requisite and pre-calculus/trigonometry is a pre-pre-requisite, then functions and their graphs are a pre-pre-pre-requisite! But... that doesn't mean that most of you are sufficiently good at dealing with these. Recognition of basic types of functions is crucial for being able to handle material at the pace and level you will need. So is the ability to go back and forth between analytic expressions for functions and their graphs. So is number sense: knowing approximate values without stopping for a detailed calculuation.

Because of the preliminary nature of this material, I am not going to write comprehensive notes on it. Instead, I will assign some online homework to make sure you are where you need to be and will refer you to sections of the textbook to brush up. Here are the key concepts and vocabulary from Section 1.1 of the textbook. Know these!

- domain
- range
- notation for piecewise defintions
- absolute value function
- greatest integer function
- increasing function
- decreasing function
- even function
- odd function

Suggested reading if any of this is unfamilair: Sections 1.3 and 1.6 of Hughes-Hallett et al., *Calculus, 5th edition.*

If *too* much of this is unfamiliar, you may be in the wrong course!

1.1 Graphing

Begin by reading Section 1.2 of Thomas, paying particular attention to the following topics: composition of functions; shifting a graph; scaling a graph; reflecting a graph and reflectional symmetry. Also please skim Sections 1.3 and 1.4, though we will not be emphasizing these.

Tips on graphing an unfamiliar function, f

(i) Is the domain all real numbers, or if not, what is it? If the function has a piecewise definition, try drawing each piece separately.

(ii) Is there an obvious symmetry? If $f(-x) = f(x)$ then f is even and there is a symmetry about the y-axis. If $f(-x) = -f(x)$ then f is odd and there is 180-degree rotational symmetry about the origin.

(iii) Are there discontinuities, and if so where? Are there asymptotes?

(iv) Try values of the function near the discontinuities to get an idea of the shape – these are particularly important places.

(v) Try computing some easy points. Often $f(0)$ or $f(1)$ is easy to compute. Trig functions are easily evaluated at certain multiples of π.

(vi) Where is f positive? Where is f increasing (where is $f' > 0$)? Where is f concave upward ($f'' > 0$), versus downward ($f'' < 0$)?

(vii) Where are the maxima and minima of f and what are its values there?

(viii) Is f defined everywhere? If not, what is the domain?

(ix) What does f do as $x \to \infty$ or $x \to -\infty$?

(x) Is there a function you understand better than f which is close enough to f that their graphs look similar?

(xi) Is f periodic? Most combinations of trig functions will be periodic.

1.2 Proportionality, units and applications

Another skill most students need practice with is writing formulas for functions given by verbal descriptions. For example, knowing that an inch is 2.54 centimeters, if $f(x)$ is the mass of a bug x centimeters long, what function represents the mass of a bug x inches long?

(a) $2.54f(x)$

(b) $f(x)/2.54$

(c) $f(2.54x)$

(d) $f(x/2.54)$

It helps to think about all such problems in units. Although inches are bigger than centimeters by a factor of 2.54, numbers giving lengths in inches are *less than* numbers giving lengths in centimeters by exactly this same factor. Writing this in units prevents you from making a mistake:

$$x \text{ cm} \times \frac{1 \text{ in}}{2.54 \text{ cm}} = \frac{x}{2.54} \text{ in}\,.$$

This shows that replacing x by $x/2.54$ converts the measurement, and therefore (d) is the correct answer. OK, maybe that was too easy for you, but when the problems get more complicated, it really helps to do this.

Some more helpful facts about units are as follows.

1. You can't add or subtract quantities unless they have the same units.

2. Multiplying (resp. dividing) quantities multiplies (resp. divides) the units.

3. Taking a power raises theunits to that power. Most functions other than powers require unitless quantites for their input. For example, in a formula $y = e^{***}$ the quantity *** must be unitless. The same is true of logarithms and trig functions: their arguments (inputs) are always unitless.

4. Units tell you how a quantity transforms under scale changes. For example a square inch is $2,54^2$ times as big as a square centimeter, and a Netwon (kilogram meter per second squared) is 10^5 dynes (gram centimeter per second squared).

Often what we can easily tell about a function is that it is proportional to some combination of other quantities, where the **constant of proportionality** may or may not be known, or may vary from one version of the problem to another.

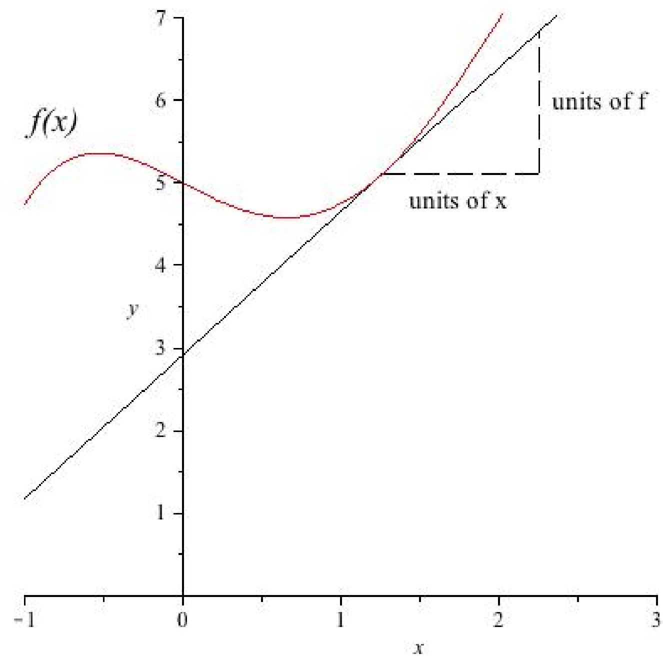
EXAMPLE: if the monetization of a social networking app is proportional to the square of the number of subscribers (this representing perhaps the amount of messaging going on) then one might write $M = kN^2$ where M is monetization, N is number of subscribers and k is the constant of proportionality. You should always give units for such constants. They can be deduced from the units of everything else. The units of N are people and the units of M are dollars, so k is in dollars per square person. You can write it: k \$/person^2.

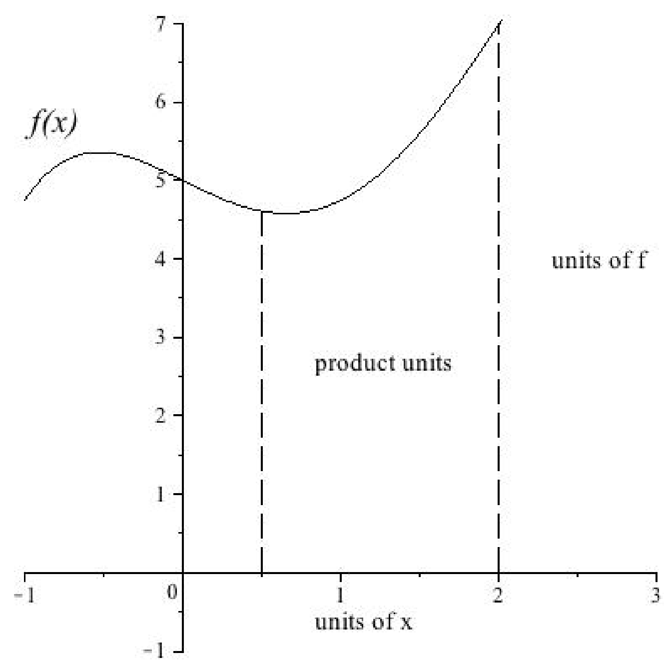
EXAMPLE: The present value under constant discounting is given by $V(t) = V_0 e^{-\alpha t}$ where V_0 is the initial value and α is the discount rate. What are the units of α? They have to be inverse time units because αt must be unitless. A typical discount rate is 2% per year. You could say that as "0.02 inverse years."

Often quantities are measured as proportions. For example, the proportional increase in sales is the change is sales divided by sales. In an equation: the proportional increase in S is $\Delta S/S$. Here, ΔS is the difference between the new and old values of S. You can subtract because both have the same units (sales), so ΔS has units of sales as well. That makes the proportional increase unitless. In fact proportions are always unitless.

Percentage increases are always unitless. In fact they are proportional increases multiplied by 100. Thus if the proportional increase is 0.0183, the percentage increase is 18.3%. In this class we aren't going to be picky about proportion versus percentage. If you say the percentage increase is 0.183 or the proportional change is 18.3%, everyone will know exactly what you mean. But you may as well be precise.

One last thing about units (really should have been point number 5 above) is how they behave under differentiation and integration. The derivative $(d/dx)f$ has units of f divided by units of x. You can see this easily on the graph because it's a limit of rise over run where rise has units of f and run has units of x. Likewise, $\int f(x)\,dx$ has units of f times units of x. Again you can see it from the picture, because the integral is an area under a graph where the y-axis has units of f and the x-axis has units of, well, x.


f(x)
units of f
units of x
y
x


f(x)
units of f
product units
units of x

Inverse functions

You can read about inverse functions is Section 1.6 of the textbook. The concept appears to be harder than most people realize. For example, last year I gave a problem to compute a formula for the inverse function to $\sinh(x)$, the hyperbolic sine function. This was already not easy (only half the students got it) but then I asked them to find a number u such that $\sinh(u) = 1$. Almost no one got it, despite that fact that this was supposed to be the easy part! They just needed to plug in 1 to their inverse sinh function. By definition, $\sinh^{-1}(1)$ is a number u such that $\sinh(u) = 1$. The moral of the story is, don't lose sight of the meaning of an inverse function when doing computations with them.

The textbook tells you how to compute them and also some things to watch out for:

- When does an inverse function exist?
- What are the domain and range of the inverse function?

When a function is not one to one, you can define an inverse if you restrict the range of the inverse. There is a standard way this is done with inverse trig functions; please read it on page 48. Note: the standard inverse trig functions have names (arcsine, etc.) and notations ($\sin^{-1}$ and so forth). Note also, that the notation f^{-1} for the inverse function to f is TERRIBLE. It is the same as (and therefore confusable with) the notation of the reciprocal of f. How stupid is that? But it's widely accepted, so we're stuck with it. Another example of a standard choice of inverse function is the inverse of the squaring function.

> Think: how is the squaring funtion not one to one, what is the name of its standard inverse, and what choice is made to remedy its lack of being one-to-one?

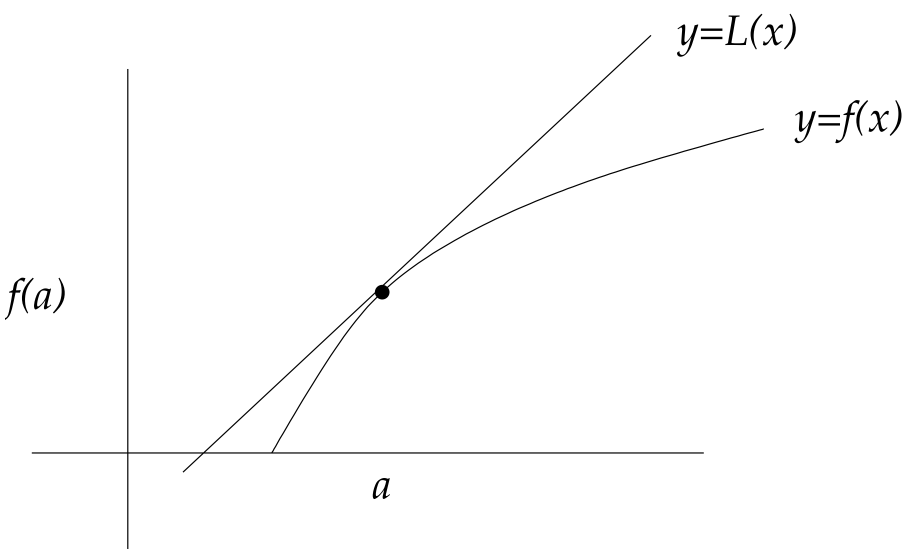
Lastly, think about the units of an inverse function. If f takes units of x as input and produces units of y, then to answer the question "f of what is equal to y?" you need to input a quantity in units of y and answer in units of x. In other words, the input and output units are switched.

1.3 Estimating and bounding

Estimating is non-rigorous. We want to understand a quantity $f(x)$ that is hard to compute, but we can compute a quantity $\tilde{f}(x)$ that is near $f(x)$. It's a little subjective to say $\tilde{f}(x) \approx f(x)$, if we can't say precisely what is meant by the symbol $\approx$, but it is very useful nonetheless. One of the most common methods of approximation is the linear approximation via the derivative. This is discussed at length in Section 3.11, too much length actually. The introduce the term *differential* which we won't use. We will however use the term *linearization*:

> **Definition:** The linearization of a differentiable function f at the point a is the function $\ell(x) = f(a) + (x - a)f'(a)$ (see page 203 of the text). In pictures $\ell(x)$ is the function whose graph is a line tangent to the graph of f at the point $(a, f(a))$.

How close is $\ell(x)$ to $f(x)$ when x is close to a? Obviously it depends how close x is to a. In the middle of the course, when we study Taylor polynomials, we'll see that $\ell(x) - f(x)$ is roughly a constant times $(x - a)^2$. Squaring makes small numbers even smaller, so when x is within 0.01 of a, then $\ell(x)$ should be within a few ten-thousandths of $f(a)$. We can't get more precise than this at present, but it's good to keep in mind.



Bounding

Bounding is rigorous. To get an upper bound on $f(x)$ means to find a quantity $U(x)$ that you understand better than $f(x)$ for which you can prove that $U(x) \geq f(x)$. A lower bound is a quantity $L(x)$ that you understand better than $f(x)$ and that you can prove to satisfy $L(x) \leq f(x)$. If you have both a lower and upper bound, then $f(x)$ is stuck for certain in the interval $[L(x), U(x)]$. It should be obvious that an upper bound is better the smaller it is. Similarly, a lower bound is better the larger it is.

In a way, though, bounding is harder than estimating because there is no one correct bound (well there's no one correct estimate either, but we usually a particular estimate we're told to use, such as a linear estimate). Two ways we typically find bounds are as follows.

First, if f is monotone increasing then an easy upper bound for $f(x)$ is $f(u)$ for any $u \geq x$ for which we can compute $f(u)$. Similarly an easy lower bound is $f(v)$ for any $v \leq x$ for which we can compute $f(v)$. If f is monotone decreasing, you can swap the roles of u and v in finding upper and lower bounds. There are even stupider useful bounds, such as $f(x) \leq C$ if f is a function that never gets above C.

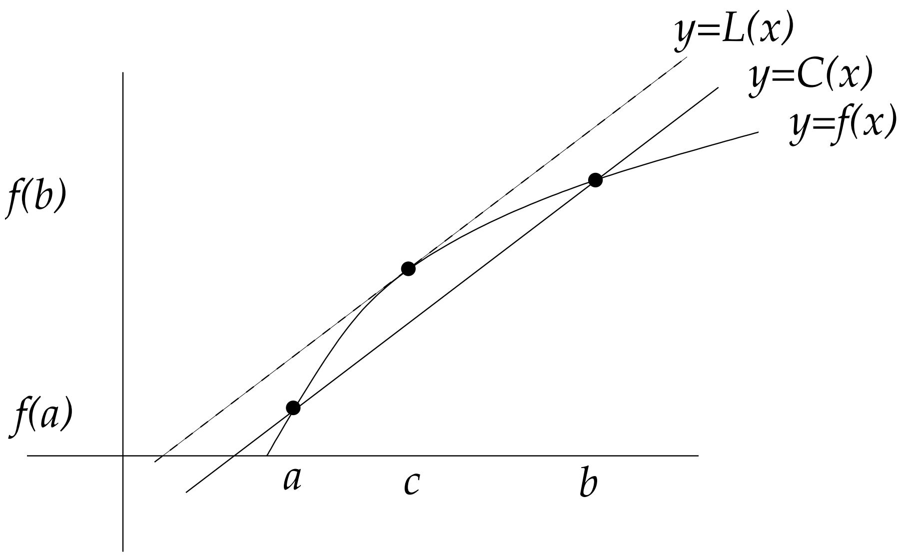
EXAMPLE: Suppose $f(x) = \sin(1)$. The easiest upper and lower bounds are 1 and -1 respectively because sin never goes aboe 1 or below -1. A better lower bound is 0 because $\sin(x)$ remains positive until $x = \pi/2$ and obviously $1 < \pi/2$. You might in fact recall that one radian is just a bit under $60°$, meaning that $\sin(60°) = \sqrt{3}/2 \approx 0.0866\ldots$ is an upper bound for $\sin(1)$. Computing more carefully, we find that a radian is also less than $58°$. Is $\sin(58°)$ a better upper bound? Probably not because we don't know how to calculate it, so it's not a quantity we understand better. Of course is we had an old-fashioned table of sines, and all we can remember about one radian is that it is between $57°$ and $58°$, then $\sin(58°)$ is not only an upper bound but the best one we have.

Concavity

A more subtle bound come when f is know to be concave upward or downward in some region. By definition, a concave upward function lies below its chords and a concave downward function lies above its chords.

Concavity upward (resp. downward) is easy to test: a function f is concave upward wherever $f'' > 0$ and concave downward wherever $f'' < 0$. If one of these holds over an interval $[a, b]$ then for x in the interval $[a, b]$ you can tell whether $f(x)$ is greater or less than the chord approximation

$$C(x) = f(a) + \frac{x-a}{b-a}(f(b) - f(a)).$$



In the figure, the function $f(x)$ is concave down. as long as x is in the interval $[a, b]$, we are guaranteed to have $C(x) \leq f(x)$. On the other hand, when $f'' < 0$ on an interval, the function always lies below the tangent line. Therefore $L(x)$ is an upper bound for $f(x)$ when $x \in [a, b]$ no matter which point $c \in [a, b]$ at which we choose to take the linear approximation.

Example: The function $\tan x$ is concave upward on $[0, \pi/2)$. That means that the tangent line to $\tan x$ anywhere in that interval will be a lower bound for $\tan x$ on the interval. The easiest place to compute the slope of $\tan x$ is at $x = 0$, where thederivative is $\sec^2(0) = 1$. The tangent line at $(0, 0)$ is therefore $y = x$. This gives the lower bound $\tan x \geq x$. This is in fact VERY close when x is near zero because tan has a point of inflection there (the tangent line pases through the curve, which is particularly flat).

1.4 Limits

You might not think limits would show up in a calculus course oriented toward application. Wrong! There are a lot of reasons why you need to understand the basic of limits. You should know these reasons, so here they are.

1. The definition of derivative (instantaneous rate of change) is a limit.‘
2. The exponential function is defined by a limit.
3. Continuous compounding is a limit.
4. Limits are needed to understand improper integrals, such as the integrals of probability densities.
5. Infinite series, which we will discuss briefly, require limits.
6. Discussing how a function behaves "in the long run" is really about limits.

I would like you to understand limits in four ways:

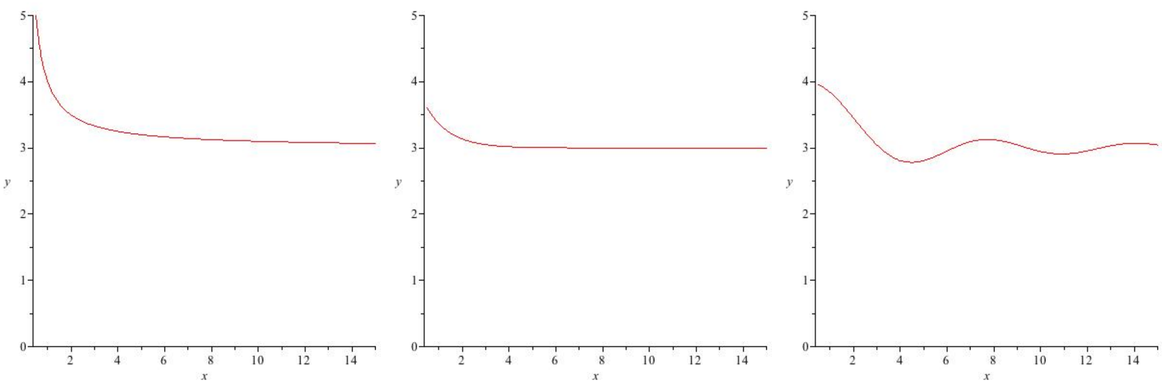
Intuitive
Pictorial
Formal
Computational

The book does a pretty good job on these, but most students do not learn limits all that well from a book so I am going to repeat some of that in these notes and in class. But please do read Section 2.2, 2.3 and 2.4.

Intuitive: The limit as $x \to a$ of $f(x)$ is the value (if any) that $f(x)$ gets close to when x gets close to (but does not equal) a. This is denoted $\lim_{x\to a} f(x)$. If we only let x approach a from one side, say from the right, we get the one-sided limit $\lim_{x\to a^+} f(x)$.

Please observe the syntax: If I tell you a function f and a value a then the expression $\lim_{x\to a} f(x)$ takes on a value (perhaps "undefined" but nonetheless a value). The variable x is a bound or "dummy" variable; it does not have a value in the expression and does not appear in the answer; it stands for a continuum of possible values approaching a.

Pictorial: if the graph of f appears to zero in on a point (a, b) as the x-coordinate gets closer to a, then that is the limit (even if the actual point (a, b) is not on the graph). Look at Example 2 on page 67 to see what I mean about (a, b) not needing to be on the graph. We can take limits at infinity as well as at a finite number. The limit as $x \to \infty$ is particularly easy visually: if $f(x)$ gets close to a number C as $x \to \infty$ then f will have a horizontal asymptote at height C (if you allow the function to possibly cross the line and double back, and still call it an asymptote). Thus $3 + \frac{1}{x}$, $3e^{-x}$ and $3 + \frac{\sin x}{x}$ all have limit 3 as $x \to \infty$.



Formal: The precise definition of a limit is found in Section 2.3. An informal poll of last semester's students showed that zero out of 45 remembered covering this in their previous course (despite the fact it was on most syllabi). The good news is, we don't have to spend a lot of time on it. However, you should see it at least once, enough that you grasp it.

The formal definition makes the intuitive definition precise. The intuitive definition is that $\lim_{x \to a} f(x) = L$ if $f(x)$ gets close to L when x gets close to a. The formal definition formalizes "gets close to" and "when". The formal definition is

> $\lim_{x \to a} f(x) = L$ if for any small tolerance ε in the y value there is a corresponding small tolerance δ in the x value such that trapping the x value in the interval $[a - \delta, a + \delta]$ (but x is not allowed to equal a) forces the y value to be trapped in the interval $[L - \varepsilon, L + \varepsilon]$. In symbols:
>
> $$0 < |x - a| < \delta \Longrightarrow |f(x) - L| < \varepsilon\,.$$

Limits at infinity. The formal definition of limit is a little different when the limit is taken at infinity. We say $\lim_{x\to\infty} f(x) = L$ if you can trap the y value within ε of L by trapping the x value above some number M. In symbols, for evey $\varepsilon > 0$ there has to be an M such that

$$x > M \Longrightarrow |f(x) - L| < \varepsilon \,.$$

Similarly, $\lim_{x\to-\infty} f(x) = L$ if for any ε you can trap y in $[L - \varepsilon, L + \varepsilon]$ by trapping x below some value $-M$:

$$x < -M \Longrightarrow |f(x) - L| < \varepsilon \,.$$

Limits of infinity. I hope you've noticed that the statement that $f(x)$ has a limit as $x \to a$ is really the statement that $\lim_{x\to a} f(x) = L$ for some real number L. The other possibility is that the limit does not exist, for which you are free to use the abbreviation DNE. This can happen because the value becomes infinite or because it has a jump, or because it is too wiggly and never settles down. Under certain conditions we say the limit is $+\infty$. NOTE: THIS IS LINGUISTAICALLY VERY MISLEADING. It is a special case of DNE. Thus we can say simultaneoulsy that the limit is $+\infty$ and that the limit DNE.

To be precise, we say that $\lim_{x\to a} f(x) = +\infty$ if for any M we can trap the y value above M by trapping the x value near enough (but not equal to) a. Formally, $\lim_{x\to a} f(x) = \infty$ if for every M there is a $\delta > 0$ such that

$$0 < |x - a| < \delta \Rightarrow f(x) > M \,.$$

You can have a limit at $\pm\infty$ of $\pm\infty$. For example $\lim_{x\to\infty} f(x) = -\infty$ when for every M there is a constant B (I used B because we already used M) such that whenever $x > B$ we have $f(x) < M$.

One-sided limits. I hope you read what's in the book about one-sided limits. We say $\lim_{x\to a^+} f(x) = L$ if you can trap $f(x)$ in any chosen interval $[L - \varepsilon, L + \varepsilon]$ by trapping x in an interval $(a, a + \delta)$ for suitably chosen δ. The intuition is that $f(x)$ gets near L when x gets near a approching from the right (on the number line) which is also called approaching from above (from higher numbers). Note again that you don't need to look at the value of f at a itself or at any point to the left of a. In fact the function doesn't have to be defined at these places, just on an interval (a, b) for some $b > a$.

Computational: There are five theorems in Section 2.2 and you should know them all. I hope they seem intuitive to you but they may not. They are not too difficult however. Next, skip ahead to Section 4.5, page 256, and read L'Hôpital's rule. Actually you need to read the whole section: the next page tells you how to iterate L'Hôpital's rule and the page after that tells you how to deal with indeterminate forms other than 0/0. L'Hôpital's rule is useful and it is also pretty easy to use. The one thing you need to be careful of is trying to apply it when you don't have an indeterminate form to begin with. I put a bunch of questions on this in the prehomework, which you can try once you've reviewed Section 4.5 of the text. Finally, here are two tricks that come in handy.

(*i*) When trying to evaluate the difference between two square roots that appear close to each other, try multiplying and dividing by their sum, for example, to evaluate

$$\sqrt{x+1}-\sqrt{x}\ \frac{(\sqrt{x+1}+\sqrt{x})}{\sqrt{x+1}+\sqrt{x}} = \frac{1}{\sqrt{x+1}+\sqrt{x}}$$

which is quite clearly a little less than $1/(2\sqrt{x})$.

(*ii*) When evaluating a limit with a power in it, try taking logs. If you can find the limit of the log, then exponentiate to get the limit of the original expression.

Examples:

(i) $\lim_{x\to\infty} \frac{x^2+3x}{2x^2-1} = \frac{1}{2}$ because we can divide by the highest power of x and obtain $\lim_{x\to\infty} \frac{1+3/x}{2-1/x^2}$. The limit of the ratio is the ratio of the limits when the individual limits exist, which the do in this case: the numerator has limit 1 and the denominator has limit 2.

(ii) $\lim_{x\to 3} \frac{x-3}{x^2-9} = \frac{1}{6}$ because you can factor $x-3$ out of the numerator and denominator. You obtain $\frac{1}{x+3}$ whenever $x \neq 3$, which has the value $\frac{1}{6}$ at $x=3$.

(iii) To find $\lim_{x\to 0}(1+2x)^{1/(3x)}$ we first find the limit of the natural log: $\lim_{x\to 0}\ln[(1+2x)^{1/(3x)}] = \lim_{x\to 0}\ln(1+2x)/(3x) = \lim_{x\to 0}\frac{2/(1+2x)}{3}$ by L'Hôpital's rule. This evaluates to 2/3 at $x=0$. The origin limit is therefore $e^{2/3}$.

Last remarks on limits:

Continuity: A function f is continuous at a point a if it has a (honest, two-sided) limit at a and this limit is equal to the function value at a (in other words, now we do require the point $(a, f(a))$ to be on the graph along with the other nearby values of f. Continuity is important later because it comes up in the hypotheses of theorems. You should read Section 2.5 and decide whether there is anything in there that is not intuitively obvious.

Limit of a sequence: The limit of a sequence $a_1, a_2, a_3, \ldots$ is nearly the same definition as $\lim_{x\to\infty} f(x)$ except that instead of a function f defined for every real value we have a term a_n that is defined only for integer values of n. Nevertheless, we say $\lim_{n\to\infty} a_n = L$ if and only if the terms of the sequence become arbitrarily close to L as n gets bigger.