HW 7 Solutions

1. (a) Converges, since it's a geometric sequence with base < 1.

$$\sum_{n=1}^{\infty} \frac{\pi}{7^n} = \pi \cdot \frac{1}{1-\frac{1}{7}} = \frac{7\pi}{6}$$

(b) Converges reason as above.

$$\sum \frac{3^n}{10^n} = \lim_{n\to\infty} \frac{1-\left(\frac{3}{10}\right)^{n+1}}{1-\frac{3}{10}} = \frac{1}{1-\frac{3}{10}} = \frac{10}{7}.$$

(c) Converges, since

$$\sum_{n=1}^{\infty} \frac{1}{\ln(n+2)} - \frac{1}{\ln(n+1)} = \lim_{n\to\infty} \sum_{i=1}^{n} \frac{1}{\ln(i+2)} - \frac{1}{\ln(i+1)}$$

$$= \lim_{n\to\infty} \left(\underbrace{\frac{1}{\ln(n+2)}}_{\to 0} - \frac{1}{\ln(2)} \right)$$

$$= -\frac{1}{\ln(2)}$$

(d) Same telescoping argument as (c).

$$\sum_{n=1}^{\infty} \frac{2}{n^2} - \frac{2}{(n+1)^2} = \lim_{n\to\infty} \left(\frac{2}{1} - \frac{2}{(n+1)^2} \right) = 2.$$

(e) since ~~an~~ $\lim_{n\to\infty} a_n = 1$. diverges

2. (a) Consider $f(x) = \frac{2}{x^2+4}$

it's positive, mono decreasing,

$$\sum_{n=1}^{\infty} \frac{2}{n^2+4} \leq \int_0^{\infty} f(x)\,dx < +\infty$$

since $\lim_{x \to +\infty} \frac{f(x)}{\frac{1}{x^2}} = 2$. and

$$\int_1^{\infty} \frac{1}{x^2}\,dx < +\infty$$

here I starts from 1 to avoid the singularity of $\frac{1}{x^2}$ at $x=0$.

Note that $f(x) = \frac{1}{x^2+4}$ does not have singularity at $x=0$.

Note: You can also compute $\int_0^{\infty} \frac{2}{x^2+4}\,dx$ directly.

2 (b) Consider $f(x) = \frac{1}{x \ln x}$. it's positive, decreasing.

Now that $\sum_{n=2}^{\infty} \frac{1}{n \ln n} \geq \int_2^{\infty} \frac{1}{x \ln x}\,dx$

$$= \ln(\ln x)\Big|_2^{\infty}$$

$$= +\infty.$$

so diverges.

(c) Let $f(x) = \frac{\ln x}{x^2}$.

$$f'(x) = \frac{\frac{1}{x} \cdot x^2 - 2x\ln x}{x^2} = \frac{x - 2x\ln x}{x^2} < 0$$

when $x \geq e$. So to be rigorous we use integral test starting from $x = e$.. (Because since $x = e$ $f(x)$ is decreasing).

$$\sum_{n=1}^{\infty} \frac{\ln n}{n^2} = \underbrace{\frac{0}{1^2} + \frac{\ln 2}{2^2} + \frac{\ln 3}{3^2}}_{\text{finite.}} + \sum_{n=4}^{\infty} \frac{\ln n}{n^2}$$

$$\leq \text{finite} + \int_3^{\infty} \frac{\ln x}{x^2}\, dx$$

$$\leq \text{finite} \quad \left(-\frac{\ln x}{x} + \frac{1}{x^2}\right)\Big|_3^{\infty} < +\infty.$$

3 (a) $\lim_{n \to +\infty} \frac{\frac{2^n}{3^{n+1}}}{\frac{2^n}{3^n}} = 1.$

Since $\sum \frac{2^n}{3^n} < +\infty \xRightarrow{\text{comparison test}} \sum \frac{2^n}{3^{n+1}} < +\infty$

(b) $\lim_{n \to +\infty} \frac{\frac{n+1}{n^4 - n^2 + 1}}{\frac{1}{n^3}} = \lim_{n \to \infty} \frac{(n+1) \cdot n^3}{n^4 - n^2 + 1} = 1.$

As a result, $\sum_{n=1}^{\infty} \frac{1}{n^3} < +\infty \Rightarrow \sum_{n=1}^{\infty} \frac{n+1}{n^4 - n^2 + 1} < +\infty$