Problem 1

(a) $\frac{dy}{dx} = xy$

$$\frac{dy}{y} = x\,dx$$

integral $\rightsquigarrow$ $\ln y = \frac{x^2}{2} + C$, $y = C_0 e^{\frac{x^2}{2}}$

(here $C_0 = e^C$. but it's just a constant)

(b) $x\frac{dy}{dx} = y + x^2 \sin x$

write it as $\frac{dy}{dx} - \frac{1}{x}y = x\sin x$

then $p(x) = -\frac{1}{x}$. $Q(x) = x\sin x$

$\int p(x)\,dx = -\ln|x|$, so $e^{\int p(x)dx} = \frac{1}{|x|}$

the solution would be

$$\frac{\int e^{\int p(x)dx} \cdot Q(x)\, dx}{e^{\int p(x)dx}}$$

$$= \frac{\int \frac{1}{|x|} \cdot x\sin x\, dx}{\frac{1}{x}} = \frac{\pm\cos x + C}{\frac{1}{x}} = x(\pm\cos x + C)$$

here the sign depends on the sign of x

(c)

$$(1+x^2)\frac{dy}{dx} = 2y$$

Then $$\frac{dy}{2y} = \frac{dx}{1+x^2}$$

integral $\rightsquigarrow$ $$\frac{1}{2}\ln|y| = \arctan x + C$$

$$|y| = C_0 e^{2\arctan x}$$

so $$y = \pm C_0 e^{2\arctan x}$$

2. The equation is

$$\frac{dy}{dt} = 0 - \frac{10y}{50+2t},$$ here y is the weight of salt.

write it as

$$\frac{dy}{10y} = -\frac{dt}{50+2t}$$

$$\frac{1}{10}\ln y = -\frac{1}{2}\ln(50+2t) + C$$

i.e. $y = e^{c}(50+2t)^{-5} = C_0 (50+2t)^{-5}$

when $t=0$, $y=10$.

$\rightsquigarrow$ $C_0 = 10 \cdot 50^{5}$

so $y = 10 \cdot 50^{5} (50+2t)^{-5}$

3 $\frac{dy}{dt} = ry\left(1-\frac{y}{K}\right)$

$\frac{dy}{y\left(1-\frac{y}{K}\right)} = r\,dt$

$LHS = \int \frac{\left(1-\frac{y}{K}\right)+\frac{y}{K}}{y\left(1-\frac{y}{K}\right)} dy = \int \left(\frac{1}{y} + \frac{1}{K}\frac{1}{1-\frac{y}{K}}\right) dy$

$= \ln y - \ln\left(1-\frac{y}{K}\right) = \ln \frac{y}{1-\frac{y}{K}}$

$RHS = rt + C$

So $\ln \frac{y}{1-\frac{y}{K}} = rt + C$

$$\frac{y}{1-\frac{y}{K}} = \underbrace{C_0 e^{rt}}_{=A} . \quad C_0 = e^C$$

$$\rightsquigarrow \quad y = A\left(1-\frac{y}{K}\right)$$

$$y\left(1+\frac{A}{K}\right) = A \qquad y = \frac{A}{1+\frac{A}{K}}$$

$$= \frac{C_0 e^{rt}}{1+\frac{C_0 e^{rt}}{K}}$$

(b) For exponential model,

$y = y_0 \cdot e^{rt}$.

when $t=0$, $y_0 = 76$

$t=50$, $y_0 \cdot e^{50r} = 150$

$r = \frac{1}{50} \ln \frac{150}{76}$, or just $e^{50r} = \frac{150}{76}$

$\rightsquigarrow$ for $t=100$, $y = 76 \cdot e^{100r} = 76\left(\frac{150}{76}\right)^2$

≈ 296

for $t=150$, $y = 76\left(\frac{150}{76}\right)^3 \approx 584$

For logistic model, i.e.

$$y = \frac{C_0 e^{rt}}{1 + \frac{C_0 e^{rt}}{K}} = \frac{1}{\frac{1}{K} + \frac{1}{C_0 e^{rt}}}$$

plug in $t=0$, $y(0) = 76$

$t = 50$, $y(50) = 150$

$t = 100$, $y(100) = 281$

$\rightsquigarrow$ get $K \approx 1421$. $r \approx -0.0147$.

$C_0 \approx 76$

$$y(150) = \frac{1}{\frac{1}{1421} + \frac{1}{76 e^{-0.0147 \cdot 150}}}$$

$$\approx 483$$