Lecture 9

1. Simplex method

Let us solve a slightly larger LP, similar to your Project #2, using the simplex method.

$$\max \quad x_1 + 2x_2 + x_3 + 4x_4 \quad \text{s.t.} \quad \begin{aligned} &3x_1 + 2x_2 + x_3 + x_4 \leq 11 \\ &x_1 + x_3 + 5x_4 \leq 5 \\ &x_1 + x_2 + x_4 \leq 3 \\ &x_2 \leq 2 \\ &x \geq 0 \end{aligned}$$

Adding slack variables $x_5, \dots, x_8$, we arrive at the following initial tableau:

(1)

	x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_8	
x_5	3	2	1	1	1	0	0	0	11
x_6	1	0	1	5	0	1	0	0	5
x_7	1	1	0	1	0	0	1	0	3
x_8	0	1	0	0	0	0	0	1	2
z	-1	-2	-1	-4	0	0	0	0	0

The corresponding basic feasible solution is $x = (0, 0, 0, 0, 11, 5, 3, 2)$, and the current value of the target function is $z = 0$.

Since we are seeking to *maximize*, we select entering variables among those with *negative* coefficient in the target row. (If we were seeking to *minimize*, we would select entering variables among those with *positive* coefficients.) Let us select x_1 as entering variable, and compute the corresponding θ-ratios:

$$\theta(x_5) = \tfrac{11}{3}, \quad \theta(x_6) = 5, \quad \theta(x_7) = 3.$$

Note that we skipped x_8 since its coefficient in the column of the entering variable x_1 is 0, so we would not be able to pivot using this entry.

Exercise 1. Explain the above, i.e., why $\{1, 5, 6, 7\}$ is not a feasible basis for the above LP.

Next, we select the departing variable with most stringent constraint, i.e., smallest θ-ratio, which is x_7. Performing the row operations we arrive at the next tableau:

(2)

	x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_8	
x_5	0	-1	1	-2	1	0	-3	0	2
x_6	0	-1	1	4	0	1	-1	0	2
x_1	1	1	0	1	0	0	1	0	3
x_8	0	1	0	0	0	0	0	1	2
z	0	-1	-1	-3	0	0	1	0	3

Exercise 2. Finish solving the LP above.

Solution to Exercise 2. Maximum is $z = 9$, achieved at $x = (1, 2, 4, 0)$, see `lecture9.nb`.

Several *pivot rules* can be followed in the implementations of the simplex algorithm, e.g.:

(i) **Largest coefficient.** Among improving variables, choose entering variable with largest (signed) coefficient in the objective row.

(ii) **Bland's rule (lexicographic).** Among improving variables, choose entering variable with the smallest index. (This rule is known to never cycle, but can be much slower.)

(iii) **Random edge.** Among improving variables, choose entering variable uniformly at random.

Exercise 3. A small workshop manufactures three types of handmade candles: small, medium, and large candles.

- A batch of small candles takes 1 hour of labor, uses 1 lb of wax, and generates \$7 in profit.
- A batch of medium candles takes 1 hour of labor, uses 2 lb of wax, and generates \$9 in profit.
- A batch of large candles takes 1 hour of labor, uses 3 lb of wax, and generates \$10 in profit.

Each day, the workshop has 40 hours of labor available and 100 lb of wax. How many batches of each type of candle should the workshop produce in order to maximize its daily profit, given these resource limits?

Solution to Exercise 3. If the workshop produces x_1 batches of small candles, x_2 batches of medium candles and x_3 batches of large candles, then the problem is rephrased as:

$$\begin{aligned} \max \quad 7x_1 + 9x_2 + 10x_3 \quad \text{s.t.} \quad & x_1 + x_2 + x_3 \le 40 \\ & x_1 + 2x_2 + 3x_3 \le 100 \\ & x \ge 0 \end{aligned}$$

Adding slack variables, we find the equational formulation:

$$\begin{aligned} \max \quad 7x_1 + 9x_2 + 10x_3 \quad \text{s.t.} \quad & x_1 + x_2 + x_3 + x_4 = 40 \\ & x_1 + 2x_2 + 3x_3 + x_5 = 100 \\ & x \ge 0 \end{aligned}$$

Applying the simplex method (see `lecture9.nb` for details), we find an optimal solution is

$$(x_1, x_2, x_3, x_4, x_5) = (0, 20, 20, 0, 0)$$

achieving maximal optimal value 380. So the workshop should produce 20 batches of medium and 20 batches of large candles, which yields a maximum daily profit of \$380.