MAT313, Spring 2025

Practice Problems for the Final Exam

1. Find all solutions to the system of linear equations

$$\begin{cases} x+2y-z=3 \\ 2x+5y+z=12 \\ 3x+7y+2z=17 \end{cases}$$

Answer: $(x,y,z)=(-2,3,1)$.

2. Find the unique polynomial $p(t)=a_2t^2+a_1t+a_0$ with $p(1)=5/2$, $p(2)=9$, $p(4)=37$.
Answer: $p(t)=\frac{5}{2}t^2-t+1$.

3. Find all solutions to the system of linear equations

$$\begin{cases} x_1+4x_2+3x_3+2x_4=0 \\ 8x_1+16x_2+4x_3+4x_4=0 \\ 6x_1+20x_2+5x_3+17x_4=0 \end{cases}$$

Answer: $x_1=3x_4,\ x_2=-2x_4,\ x_3=x_4,\ x_4$ free

4. Find the reduced row echelon form of:

$$\begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 1 & 1 & 1 \end{bmatrix}$$

Answer:

$$\begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}$$

5. Can the vector $v=\begin{bmatrix} 4 \\ 1 \\ 4 \end{bmatrix}$ be written as a linear combination of $w_1=\begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}$ and $w_2=\begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}$?
Answer: Yes: $v=2w_1+w_2$

6. Let $T\colon \mathbb{R}^2 \to \mathbb{R}^2$ be given by $T(x,y)=(x+y,xy)$. Determine whether T is a linear transformation. If so, then:

(a) Find the matrix that represents T in the canonical basis.

(b) Is T invertible? If so, find a formula for its inverse.

Answer: Not linear due to xy term (not additive or scalar multiplicative)

7. Let $T\colon \mathbb{R}^3 \to \mathbb{R}^3$ be given by $T(x, y, z) = (z, x + y, x - y)$. Determine whether T is a linear transformation. If so, then:

   (a) Find the matrix that represents T in the canonical basis.

   (b) Is T invertible? If so, find a formula for its inverse.

   Answer: a) The matrix that represents T in the canonical basis is

$$\begin{bmatrix} 0 & 0 & 1 \\ 1 & 1 & 0 \\ 1 & -1 & 0 \end{bmatrix}$$

   b) Yes, it is invertible and T^{-1} is represented in the canonical basis by the matrix

$$\begin{bmatrix} 0 & 1/2 & 1/2 \\ 0 & 1/2 & -1/2 \\ 1 & 0 & 0 \end{bmatrix}$$

8. Let $T\colon \mathbb{R}^2 \to \mathbb{R}^3$ be a linear map, such that $T(1, 2) = (3, 6, 9)$ and $T(2, 7) = (18, -3, 1/2)$.

   (a) Find $T(0, 1)$.

   (b) What is the matrix that represents T in the canonical basis?

   (c) What are the dimensions of the subspaces $\ker T$ and $\operatorname{Im} T$?

   Answer: a) $T(0, 1) = \left(4, -5, -\frac{35}{6}\right)$,

   b) $[T] = \begin{bmatrix} -5 & 4 \\ 16 & -5 \\ \frac{62}{3} & -\frac{35}{6} \end{bmatrix}$

   c) $\dim \operatorname{Im} T = 2, \quad \dim \ker T = 0$

9. Determine if the set $\{(1, 2, 3), (2, 4, 6), (0, 1, 1)\}$ is linearly independent.
   Answer: No, the first two vectors are linearly dependent.

10. Find the value of t such that the vectors $\begin{bmatrix} t \\ t \\ 4 \end{bmatrix}$ and $\begin{bmatrix} 1 + t \\ 2t \\ 5 + 3t \end{bmatrix}$ are linearly dependent.

    Answer: $t = 1$.

11. Write a basis for the subspace of $\mathbb{R}^8$ consisting of solutions to the following system:

$$\begin{cases} x_1 + x_2 = 0 \\ x_2 + x_3 = 0 \\ x_3 + x_4 = 0 \\ x_4 + x_5 = 0 \\ x_5 + x_6 = 0 \\ x_6 + x_7 = 0 \\ x_7 + x_8 = 0 \end{cases}$$

Answer: $\{(-1, 1, -1, 1, -1, 1, -1, 1)\}$.

12. Decide if each of the following sets is linearly dependent or linearly independent:

(a) $\{(4,6), (1,1), (3,7)\}$ in $\mathbb{R}^2$
(b) $\{(1,-1), (1,1)\}$ in $\mathbb{R}^2$
(c) $\{(0,1,1), (1,0,1), (1,1,0)\}$ in $\mathbb{R}^3$
(d) $\{\left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right), \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right), \left(\begin{smallmatrix} -1 & 0 \\ 0 & 1 \end{smallmatrix}\right), \left(\begin{smallmatrix} 2 & 2 \\ 2 & 2 \end{smallmatrix}\right)\}$ in $\mathrm{Mat}_{2\times 2}(\mathbb{R})$
(e) $\{t^2 + 5t, t - 1, t, 7\}$ in $\mathbb{R}[t]_3$

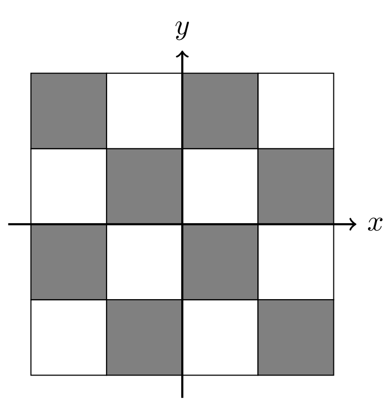
Answer: a) lin dep, b) lin indep, c) lin indep, d) lin dep, e) lin dep

13. Find the vector obtained by rotating the vector $\begin{bmatrix} 2 \\ 3 \end{bmatrix}$ by 45^o counterclockwise.

Answer:

$$\begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} 2 \\ 3 \end{bmatrix} = \begin{bmatrix} \sqrt{2} - \frac{3}{\sqrt{2}} \\ \sqrt{2} + \frac{3}{\sqrt{2}} \end{bmatrix}$$

14. Sketch the image of the checkerboard below under each of linear transformation:



(a) $T = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$

(b) $T = \begin{bmatrix} 3 & 0 \\ 0 & \frac{1}{2} \end{bmatrix}$

(c) $T = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$

15. Is $A = \begin{bmatrix} 1 & 2 \\ 3 & 6 \end{bmatrix}$ invertible? If so, find its inverse.
Answer: No, since $\det(A) = 0$.

16. Suppose that A is an invertible matrix. Is its transpose A^T also invertible? Justify.
Answer: Yes, since $\det(A^T) = \det(A)$.

17. Suppose that A is symmetric. Is A invertible? Justify.
Answer: No, e.g. $A = 0$ is symmetric but not invertible.

18. Suppose that A is symmetric. Is A^2 symmetric? Justify.
Answer: Yes: $A^T = A$ so $(A^2)^T = (AA)^T = (A^T)(A^T) = AA = A^2$.

19. Suppose that A and B are invertible. Is A^2B^2 invertible? What about $ABAB$?
Answer: Yes: $(A^2B^2)^{-1} = (B^{-1})^2(A^{-1})^2$, $(ABAB)^{-1} = B^{-1}A^{-1}B^{-1}A^{-1}$.

20. Find the rank and nullity of

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 4 & 5 & 6 \end{bmatrix}$$

Answer: Rank = 2, Nullity = 1

21. Compute the determinant of the matrices $A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 4 & 9 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 0 & 0 \\ 2 & -3 & 0 \\ 5 & 4 & 4 \end{bmatrix}$.

Answer: $\det A = 2,\ \det B = -12$

22. Find a basis for $\ker A$, where

$$A = \begin{bmatrix} 1 & 2 & -1 \\ 2 & 4 & -2 \end{bmatrix}$$

Answer: $\left\{ \begin{bmatrix} 1 \\ -1/2 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \right\}$

23. Find bases for $\ker T$ and $\operatorname{Im} T$ for each of the following linear transformations:

(a) $T\colon \mathbb{R}^2 \to \mathbb{R}^2$, $T(x,y) = (x+y, 2x+2y)$

(b) $T\colon \mathbb{R}^4 \to \mathbb{R}^2$, $T(x,y,z,w) = (x,w)$

(c) $T\colon \mathbb{R} \to \mathbb{R}^2$, $T(x) = (x, 5x)$

(d) $T\colon \mathbb{R}[t]_2 \to \mathbb{R}[t]_3$, $T(p(t)) = \int p(t)\,\mathrm{d}t$

(e) $T\colon \operatorname{Mat}_{2\times 2}(\mathbb{R}) \to \mathbb{R}^2$, $T(A) = (a_{11}, a_{22})$, where $A = \left(\begin{smallmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{smallmatrix}\right)$

Answer: a) $\ker T = \operatorname{span}\{(1,-1)\}$, $\operatorname{Im} T = \operatorname{span}\{(1,2)\}$
b) $\ker T = \operatorname{span}\{(0,1,0,0),(0,0,1,0)\}$, $\operatorname{Im} T = \operatorname{span}\{(1,0),(0,1)\}$
c) $\ker T = \{0\}$, $\operatorname{Im} T = \operatorname{span}\{(1,5)\}$
d) $\ker T = \{0\}$, $\operatorname{Im} T = \operatorname{span}\{t, t^2, t^3\}$
e) $\ker T = \left\{ \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \right\}$, $\operatorname{Im} T = \operatorname{span}\{(1,0),(0,1)\}$, note this is same as b) up to isomorphism.

24. Find the dimension of the subspace of $\mathbb{R}^5$ given by vectors of the form

$$\big(a - 3b + c,\ 2a - 6b - 2c,\ 3a - 9b + c,\ c,\ 2a - 6b + 6c\big)$$

Answer: 2

25. Find the coordinates of $v = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$ with respect to the basis $\mathcal{B} = \left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\}$.

Answer: $[v]_{\mathcal{B}} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$

26. Find the eigenvalues of the following matrices

(a) $\begin{bmatrix} 4 & 3 \\ 3 & 4 \end{bmatrix}$

(b) $\begin{bmatrix} 0 & 1 \\ 1 & 4 \end{bmatrix}$

(c) $\begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix}$

Answer:
a) $7, 1$
b) $2 \pm \sqrt{5}$
c) $2, 2, 0$

27. Find the eigenvalues and corresponding eigenspaces of

$$A = \begin{bmatrix} 4 & 1 \\ 2 & 3 \end{bmatrix}$$

Answer: Eigenvalues: $\lambda = 5, 2$. Eigenspaces: $E_5 = \text{span}\{(1,1)\}$, $E_2 = \text{span}\{(-1,2)\}$.

28. Determine whether the matrix

$$A = \begin{bmatrix} 4 & 1 \\ 0 & 4 \end{bmatrix}$$

is diagonalizable. If so, find a diagonalization.
Answer: Not diagonalizable because eigenvalue $\lambda = 4$ has algebraic multiplicity 2 but geometric multiplicity 1.

29. Determine whether the matrix

$$A = \begin{bmatrix} 6 & 2 \\ 2 & 3 \end{bmatrix}$$

is diagonalizable. If so, find a diagonalization.
Answer: Eigenvalues: $\lambda = 7, 2$. Eigenspaces: $E_7 = \text{span}\{(2,1)\}$, $E_2 = \text{span}\{(-1,2)\}$.
So $P = \begin{bmatrix} 2 & -1 \\ 1 & 2 \end{bmatrix}$ and $D = \begin{bmatrix} 7 & 0 \\ 0 & 2 \end{bmatrix}$ are such that $A = PDP^{-1}$.

30. Find A^{100} where $A = \begin{bmatrix} 4 & 3 \\ 3 & 4 \end{bmatrix}$.

Answer: $A^{100} = \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ -1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 7^{100} \end{bmatrix} \begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix} = \begin{bmatrix} \frac{7^{100}+1}{2} & \frac{7^{100}-1}{2} \\ \frac{7^{100}-1}{2} & \frac{7^{100}+1}{2} \end{bmatrix}$.

31. Apply Gram-Schmidt to find an orthonormal basis of $W = \text{span}\{(1,1,0),(1,0,1)\}$.

Answer: Orthonormal basis: $\left\{ (\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0), (\frac{1}{\sqrt{6}}, -\frac{1}{\sqrt{6}}, \sqrt{\frac{2}{3}}) \right\}$

32. Find an orthonormal basis of the subspace $\{(x,y,z) \in \mathbb{R}^3 : x + y + z = 0\}$.

Answer: Orthonormal basis: $\left\{ (\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}, 0), (\frac{1}{\sqrt{6}}, \frac{1}{\sqrt{6}}, -\sqrt{\frac{2}{3}}) \right\}$

33. Find a unit vector in $\mathbb{R}^2$ that is orthogonal to the line $y = mx + b$, where $m \neq 0$.

Answer: $v = \left(\frac{1}{\sqrt{1+\frac{1}{m^2}}}, -\frac{1}{\sqrt{1+m^2}} \right)$.

34. Find the point in the plane $x - 2y + z = 0$ which is closest to $(1, 0, 0)$.

Answer: An orthonormal basis for the plane is $\left\{\left(\frac{1}{\sqrt{2}}, 0, -\frac{1}{\sqrt{2}}\right), \left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right)\right\}$, and the orthogonal projection of $(1, 0, 0)$ on this plane is $\left(\frac{5}{6}, \frac{1}{3}, -\frac{1}{6}\right)$.

35. Let A be a 3×3 matrix with eigenvalues $2, 5, 10$. Is A invertible? If so, compute the eigenvalues of A^{-1}.

Answer: Yes, since $A = PDP^{-1}$ with P orthogonal and $D = \begin{bmatrix} 2 & & \\ & 5 & \\ & & 10 \end{bmatrix}$, we see that $\det A = 100$ so A is invertible. Moreover, $A^{-1} = (PDP^{-1})^{-1} = P^{-1}D^{-1}P$ so A^{-1} has eigenvalues $1/2, 1/5, 1/10$.

36. If A is diagonalizable, prove that $A + c\,\mathrm{Id}$ is also diagonalizable for any $c \in \mathbb{R}$.

Answer: Since $A = PDP^{-1}$ with P orthogonal and D diagonal, we see that

$$A + c\,\mathrm{Id} = PDP^{-1} + c\,\mathrm{Id} = PDP^{-1} + P\,c\,\mathrm{Id}\,P^{-1} = P(D + c\,\mathrm{Id})P^{-1}$$

is diagonalizable and has eigenvalues $\lambda + c$ where λ is an eigenvalue of A.

37. Does the matrix $\begin{bmatrix} 4 & 0 & -1 \\ 0 & 2 & 1 \\ 1 & -1 & 3 \end{bmatrix}$ admit an orthonormal basis of eigenvectors? Justify.

Answer: No, because it is not symmetric.

38. Find the singular values of the matrices $\begin{bmatrix} \frac{1}{2} & 0 \\ 0 & 2 \\ 1 & 1 \end{bmatrix}$ and $\begin{bmatrix} 1 & 0 \\ 0 & 4 \\ 2 & 2 \end{bmatrix}$.

Answer: The matrix A has singular values $\sigma_1(A) = \frac{\sqrt{21}}{2}$ and $\sigma_2(A) = 1$, so the matrix $2A$ has singular values $2\sigma_1(A) = \sqrt{21}$ and $2\sigma_2(A) = 2$.