Lecture 9

1. Simplex method

Let us solve a slightly larger LP, similar to your Project #2, using the simplex method.

$$\max \quad x_1 + 2x_2 + x_3 + 4x_4 \quad \text{s.t.} \quad \begin{aligned} &3x_1 + 2x_2 + x_3 + x_4 \leq 11 \\ &x_1 + x_3 + 5x_4 \leq 5 \\ &x_1 + x_2 + x_4 \leq 3 \\ &x_2 \leq 2 \\ &x \geq 0 \end{aligned}$$

Adding slack variables $x_5, \dots, x_8$, we arrive at the following initial tableau:

(1)

	x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_8	
x_5	3	2	1	1	1	0	0	0	11
x_6	1	0	1	5	0	1	0	0	5
x_7	1	1	0	1	0	0	1	0	3
x_8	0	1	0	0	0	0	0	1	2
z	-1	-2	-1	-4	0	0	0	0	0

The corresponding basic feasible solution is $x = (0, 0, 0, 0, 11, 5, 3, 2)$, and the current value of the target function is $z = 0$.

Since we are seeking to *maximize*, we select entering variables among those with *negative* coefficient in the target row. (If we were seeking to *minimize*, we would select entering variables among those with *positive* coefficients.) Let us select x_1 as entering variable, and compute the corresponding θ-ratios:

$$\theta(x_5) = \tfrac{11}{3}, \quad \theta(x_6) = 5, \quad \theta(x_7) = 3.$$

Note that we skipped x_8 since its coefficient in the column of the entering variable x_1 is 0, so we would not be able to pivot using this entry.

Exercise 1. Explain the above, i.e., why $\{1, 5, 6, 7\}$ is not a feasible basis for the above LP.

Next, we select the departing variable with most stringent constraint, i.e., smallest θ-ratio, which is x_7. Performing the row operations we arrive at the next tableau:

(2)

	x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_8	
x_5	0	-1	1	-2	1	0	-3	0	2
x_6	0	-1	1	4	0	1	-1	0	2
x_1	1	1	0	1	0	0	1	0	3
x_8	0	1	0	0	0	0	0	1	2
z	0	-1	-1	-3	0	0	1	0	3

Exercise 2. Finish solving the LP above.

Solution to Exercise 2. Maximum is $z = 9$, achieved at $x = (1, 2, 4, 0)$, see `lecture9.nb`.

Several *pivot rules* can be followed in the implementations of the simplex algorithm, e.g.:

(i) **Largest coefficient.** Among improving variables, choose entering variable with largest (signed) coefficient in the objective row.
(ii) **Bland's rule (lexicographic).** Among improving variables, choose entering variable with the smallest index. (This rule is known to never cycle, but can be much slower.)
(iii) **Random edge.** Among improving variables, choose entering variable uniformly at random.