Lecture 5

1. Extremal points

Recall that $S \subset \mathbb{R}^n$ is *convex* if given any $x, y \in S$, the line segment $(1-t)x + ty$, $0 \le t \le 1$, joining x and y lies entirely in S. A set $S \subset \mathbb{R}^n$ is *bounded* if there exists $R > 0$ such that all points in S are at distance at most R from $0 \in \mathbb{R}^n$, that is, for all $x \in S$, $\|x\| \le R$.

A point $v \in S$ in a convex set is called *extremal* if $v = (1-t)x + ty$ with $x, y \in S$ and $0 \le t \le 1$ implies that either $t = 0$ or $t = 1$. In other words, v is extremal if it *cannot* be placed in the *interior* of any line segment with endpoints in S.

Exercise 1. Determine the extremal points of the following convex sets:

(i) A bounded polyhedron $S \subset \mathbb{R}^n$
(ii) The unit ball $B = \{x \in \mathbb{R}^n : \|x\| \le 1\}$

Solution to Exercise 1. The extremal points are:

(i) vertices of S (there are only finitely many);
(ii) all the points in the boundary $\partial B = \{x \in \mathbb{R}^n : \|x\| = 1\}$ (there are infinitely many)

A *convex combination* of the points $x_1, \dots, x_r \in \mathbb{R}^n$ is any point of the form

$$c_1 x_1 + \dots + c_r x_r \in \mathbb{R}^n,$$

where $c_1, \dots, c_r \in \mathbb{R}$ satisfy $\sum_{i=1}^{r} c_i = 1$ and $c_i \ge 0$ for all $1 \le i \le r$. The set of all convex combinations of $x_1, \dots, x_r$ is called the *convex hull* of $x_1, \dots, x_r$, and denoted $\operatorname{conv}(x_1, \dots, x_r)$.

Exercise 2. Prove that $\operatorname{conv}(x_1, \dots, x_r)$ is convex.

Exercise 3. What is the convex hull of 2 points in $\mathbb{R}^n$?

Exercise 4. What is the convex hull of n points in $\mathbb{R}^2$?

The following are foundational statements that we will use but not prove. (You might want to think about how you would prove them.)

Theorem 1. *A polyhedron is bounded if and only if it does not contain a line.*

Theorem 2 (Krein-Milman, baby version)**.** *A bounded polyhedron coincides with the convex hull of its vertices (i.e., its extremal points).*

By the above, "determining" a bounded polyhedron is the same as "determining" its vertices. In order to do this using as input the description $S = \{x \in \mathbb{R}^n : Ax \le b\}$ of a polyhedron as an intersection of half-spaces $a_i^T x \le b_i$, we use the following result:

Theorem 3. *Consider the polyhedron $S = \{x \in \mathbb{R}^n : Ax \le b\}$, where A is an $m \times n$ matrix and $b \in \mathbb{R}^m$. A point $v \in S$ is a vertex of S if and only if there exist n linearly independent inequality constraints of S that hold with equality at v, i.e., there exist $i_1, \dots, i_n \in \{1, \dots, m\}$ such that $a_{i_1}^T v = b_{i_1}, \dots, a_{i_n}^T v = b_{i_n}$ and $\{a_{i_1}, \dots, a_{i_n}\}$ are linearly independent.*

The above yields a method to find all vertices of a polyhedron $S = \{x \in \mathbb{R}^n : Ax \le b\}$, namely one can proceed as follows. `For` each[1] subset $\{i_1, \dots, i_n\}$ of $\{1, \dots, m\}$, `do`:

(i) Check if $a_{i_1}, \dots, a_{i_n}$ are linearly independent (if `NO`, then `STOP`);
(ii) Compute the unique solution $v \in \mathbb{R}^n$ to $a_{i_1}^T v = b_{i_1}, \dots, a_{i_n}^T v = b_{i_n}$;
(iii) If $v \in S$, i.e., $Av \le b$, then v is a vertex. If not, then it is not a vertex.

[1]Note there are $\binom{m}{n}$ such subsets.

Running the above `for` loop through all subsets of $\{1,\dots,m\}$ and collecting the resulting vertices, one obtains the complete list of vertices of S. In particular, this proves that a polyhedron only has finitely many vertices.

Exercise 5. Find all vertices of the polyhedron $S=\{x\in\mathbb{R}^n : Ax\le b\}$ where

(i) $A=\begin{pmatrix}-1 & 0\\ 0 & -1\\ 1 & 1\end{pmatrix}$, $b=\begin{pmatrix}0\\0\\1\end{pmatrix}$

(ii) $A=\begin{pmatrix}2 & 4\\ 1 & 1\\ 0 & 1\\ 1 & -1\\ -1 & 0\\ 0 & -1\end{pmatrix}$, $b=\begin{pmatrix}12\\5\\5/2\\4\\0\\0\end{pmatrix}$. Note this polygon appeared in Lecture 3, Exercise 4.

2. Linear programs in any number of variables

A general *linear program (LP)* in n variables $x=(x_1,x_2,\dots,x_n)$ is an optimization problem of the form

$$\begin{aligned}\min \quad c_1x_1+c_2x_2+\cdots+c_nx_n \quad \text{s.t.} \quad & a_{11}\,x_1+a_{12}\,x_2+a_{1n}\,x_n\le b_1,\\ & a_{21}\,x_1+a_{22}\,x_2+a_{2n}\,x_n\le b_2,\\ & \dots\\ & a_{m1}\,x_1+a_{m2}\,x_2+a_{mn}\,x_n\le b_m.\end{aligned}\tag{1}$$

The above constraints might have been obtained from linear constraints with $\le$, $=$, or $\ge$, using the elementary tricks we discussed in lecture, and might (or might not) include the nonnegativity constraints $x_1\ge 0,\dots x_n\ge 0$. Recall that maximization problems reduce to the above as well.

Note that the above LP can be rewritten in matrix notation as

$$\min \quad c^T x \quad \text{s.t.} \quad A\,x\le b,\tag{2}$$

where $x=(x_1,x_2,\dots,x_n)\in\mathbb{R}^n$, $c=(c_1,c_2,\dots,c_n)\in\mathbb{R}^n$, $b=(b_1,\dots,b_m)\in\mathbb{R}^m$, and

$$A=\begin{pmatrix}a_{11} & a_{12} & \dots & a_{1n}\\ a_{21} & a_{22} & \dots & a_{2n}\\ \vdots & \vdots & & \vdots\\ a_{m1} & a_{m2} & \dots & a_{mn}\end{pmatrix}$$

The rows of A are denoted $a_1=(a_{11},a_{12},\dots,a_{1n})$, $\dots$, $a_m=(a_{m1},a_{m2},\dots,a_{mn})$. These conventions are as in earlier lectures; in particular, the indices have ranges $1\le i\le m$ and $1\le j\le n$. Finally, the following statement explains the relevance of extremal points.

Theorem 4. *If the optimization problem* (2) *is* feasible *and* bounded, *i.e., the polyhedron* $S=\{x\in\mathbb{R}^n : Ax\le b\}$ *is nonempty and bounded, then there exists an extremal point* $v\in S$ *which is an optimal solution.*