Lecture 24

1. Duality

1.1. **LP.** Recall from Lectures 14-15 that the dual of the primal LP

$$\min \quad c^T x \quad \text{s.t.} \quad \begin{aligned} &Ax = b, \\ &x \geq 0, \end{aligned}$$

is given by

$$\max \quad b^T y \quad \text{s.t.} \quad A^T y \leq c.$$

Feasible solutions give upper/lower bounds for the optimal value of the dual problem, since

$$c^T x - b^T y = x^T c - (Ax)^T y = x^T (c - A^T y) \geq 0.$$

Moreover, the Strong Duality Theorem ensured that if both primal and dual are feasible, then optimal solutions x_* and y_* exist and the corresponding optimal values agree, that is, $c^T x_* = b^T y_*$.

1.2. **SDP.** Given the primal SDP on the variable $X \in \mathrm{Sym}^2(\mathbb{R}^d)$,

$$\min \quad \langle C, X\rangle \quad \text{s.t.} \quad \begin{aligned} &\langle A_i, X\rangle = b_i, \;\; i = 1, \dots, m, \\ &X \succeq 0, \end{aligned}$$

the dual SDP is

$$\max \quad b^T y \quad \text{s.t.} \quad \sum_{i=1}^m y_i A_i \preceq C,$$

on the variable $y \in \mathbb{R}^m$. Similarly to LPs, feasible solutions give upper/lower bounds for the dual:

$$\langle C, X\rangle - b^T y = \langle C, X\rangle - \sum_{i=1}^m y_i \langle A_i, X\rangle = \left\langle C - \sum_{i=1}^m y_i A_i, X \right\rangle \geq 0.$$

Exercise 1. Justify the last inequality: prove that if $P, Q \in \mathrm{Sym}^2(\mathbb{R}^d)$ satisfy $P \succeq 0$ and $Q \succeq 0$, then $\langle P, Q\rangle \geq 0$ and equality holds if and only if $PQ = QP = 0$. Hint: $P = R^T R$, $Q = S^T S$.

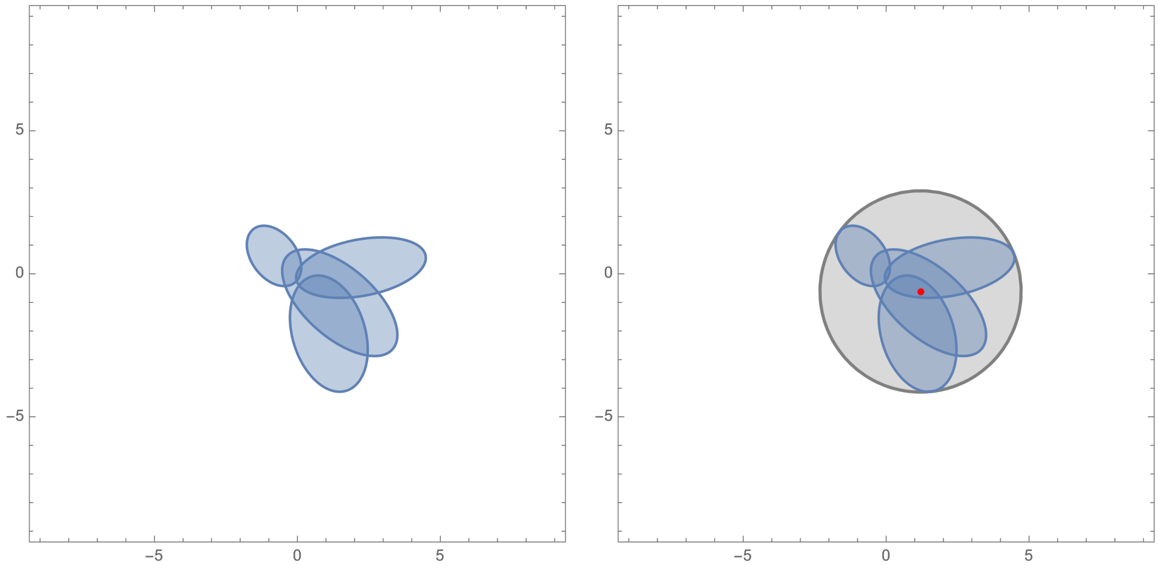
Solution to Exercise 1. Following the hint, recall that since $P, Q \in \mathrm{Sym}^2(\mathbb{R}^d)$ satisfy $P \succeq 0$ and $Q \succeq 0$, there exist $R, S \in \mathrm{Sym}^2(\mathbb{R}^d)$ such that $P = R^T R$, $Q = S^T S$. Then,

$$\begin{aligned} \langle P, Q\rangle = \operatorname{tr} PQ^T = \operatorname{tr}(R^T R)(S^T S)^T = \operatorname{tr} R^T R S^T S \\ = \operatorname{tr}(RS^T)(SR^T) = \operatorname{tr}(RS^T)(RS^T)^T = \langle RS^T, RS^T\rangle \geq 0. \end{aligned}$$

However, *strong* duality fails for SDP, see [BPT13, Ex 2.14] for an example. In order to have equality between the optimal values of primal and dual SDP, a sufficient condition is that both are *strictly feasible*, i.e., there exists $X \succ 0$ satisfying the constraints of the primal and there exists y such that $C - \sum_i y_i A_i \succ 0$ for the dual.

2. Using SDP to solve a geometric problem

Consider the problem of finding the smallest disk in $\mathbb{R}^2$ that contains a given number of ellipses.[1]



Assume the ellipses are the sublevelsets $\mathcal{E}_i = \{x \in \mathbb{R}^2 : q_i(x) \leq 0\}$ of the quadratic functions

$$q_i(x) = x^T A_i x + 2b_i^T x + c_i,$$

where $A_i \in \mathrm{Sym}^2(\mathbb{R}^2)$ is a positive-semidefinite matrix, $b_i \in \mathbb{R}^2$, $c_i \in \mathbb{R}$. We shall use the following:

Proposition 1. *The ellipse $\mathcal{E} = \{x \in \mathbb{R}^2 : q(x) \leq 0\}$ contains the ellipse $\overline{\mathcal{E}} = \{x \in \mathbb{R}^2 : \overline{q}(x) \leq 0\}$, where $q(x) = x^T A x + 2b^T x + c$ and $\overline{q}(x) = x^T \overline{A} x + 2\overline{b}^T x + \overline{c}$ if and only if there is $\tau \geq 0$ such that*[2]

$$\begin{pmatrix} A & b \\ b^T & c \end{pmatrix} \preceq \tau \begin{pmatrix} \overline{A} & \overline{b} \\ \overline{b}^T & \overline{c} \end{pmatrix}.$$

Thus, a circle $\mathcal{C} = \{x \in \mathbb{R}^2 : q_c(x) \leq 0\}$, where $q_c(x) = x^T x - 2x_c^T x + \gamma$ contains the ellipses $\mathcal{E}_i$, $i = 1, \ldots, p$, if and only if there exists $\tau_i \geq 0$ such that

$$\begin{pmatrix} \mathrm{Id} & -x_c \\ -x_c^T & \gamma \end{pmatrix} \preceq \tau_i \begin{pmatrix} A_i & b_i \\ b_i^T & c_i \end{pmatrix}, \quad i = 1, \ldots, p.$$

Exercise 2. a) Show that the radius of the circle $\mathcal{C}$ is $\sqrt{x_c^T x_c - \gamma}$.
b) Write an SDP that is equivalent to the geometric optimization problem at hand:

$$\begin{aligned} \min \quad \sqrt{x_c^T x_c - \gamma} \quad \text{s.t.} \quad & \begin{pmatrix} \mathrm{Id} & -x_c \\ -x_c^T & \gamma \end{pmatrix} \preceq \tau_i \begin{pmatrix} A_i & b_i \\ b_i^T & c_i \end{pmatrix}, \quad i = 1, \ldots, p. \\ & \tau_i \geq 0, \quad i = 1, \ldots, p. \end{aligned}$$

Solution to Exercise 2. a) The radius can be found by completing the square in $q_c(x) \leq 0$.
b) In order to minimize the radius of $\mathcal{C}$, we minimize t such that

$$\begin{pmatrix} \mathrm{Id} & x_c \\ x_c^T & t + \gamma \end{pmatrix} \succeq 0.$$

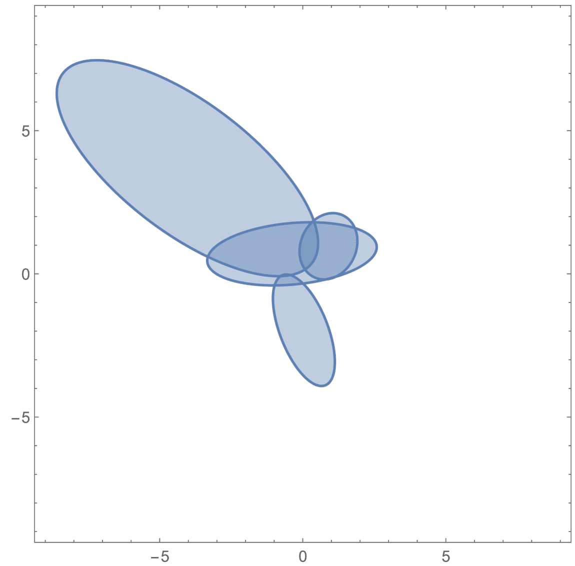
[1] This example is taken from [VB96, p. 58].

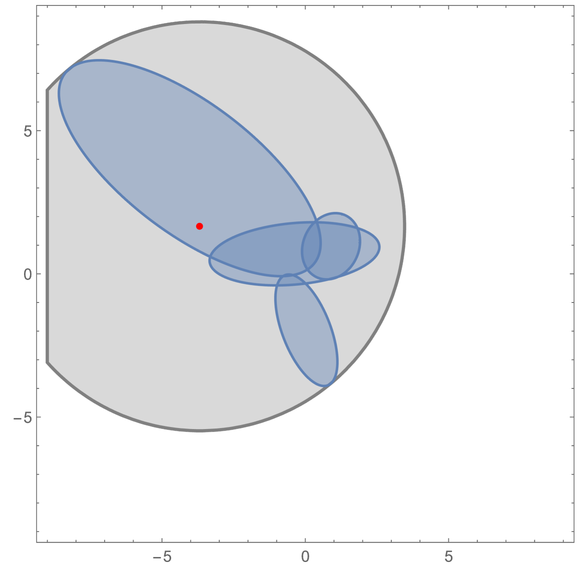
[2] We are implicitly assuming that both ellipses have nonempty interior.

Thus, we arrive at the SDP

$$\begin{aligned} \min \quad t \quad \text{s.t.} \quad & \begin{pmatrix} \mathrm{Id} & -x_c \\ -x_c^T & \gamma \end{pmatrix} \preceq \tau_i \begin{pmatrix} A_i & b_i \\ b_i^T & c_i \end{pmatrix}, \quad i = 1, \ldots, p. \\ & \tau_i \geq 0, \quad i = 1, \ldots, p. \\ & \begin{pmatrix} \mathrm{Id} & x_c \\ x_c^T & t + \gamma \end{pmatrix} \succeq 0. \end{aligned}$$

See file `lecture24.nb` for an implementation.





References

[BPT13] G. Blekherman, P. A. Parrilo, and R. R. Thomas. *Semidefinite optimization and convex algebraic geometry*, vol. 13 of MOS-SIAM Series on Optimization. Society for Industrial and Applied Mathematics (SIAM), Philadelphia, PA; Mathematical Optimization Society, Philadelphia, PA, 2013.

[VB96] L. Vandenberghe, S. Boyd. *Semidefinite Programming*, SIAM Review, Vol. 38, No. 1, 49-95, 1996. `https://web.stanford.edu/~boyd/papers/pdf/semidef_prog.pdf`