Lecture 22

1. Examples

Let us work through some examples of semidefinite programs that can be solved geometrically. Recall that the feasible set of an SDP is a spectrahedron, hence a basic closed semialgebraic set.

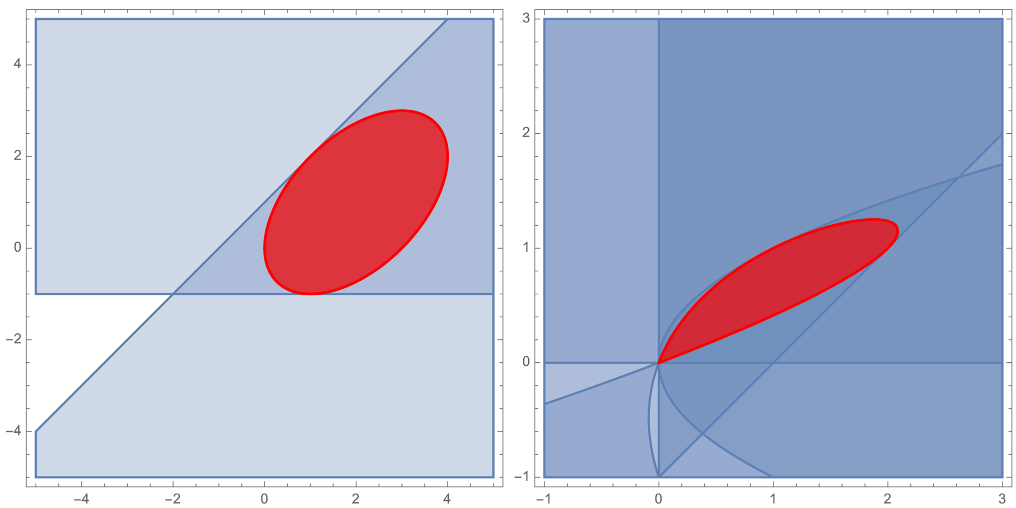
Exercise 1. Consider the spectrahedron $S \subset \mathbb{R}^2$ defined by

$$S = \left\{ (x,y) \in \mathbb{R}^2 : \begin{pmatrix} x-y+1 & x-1 \\ x-1 & y+1 \end{pmatrix} \succeq 0, \; \begin{pmatrix} y+1 & x & 1 \\ x & x & y \\ 1 & y & 1 \end{pmatrix} \succeq 0 \right\}$$

a) Write S as a basic semialgebraic set using the fewest possible polynomial inequalities.
b) Plot S and describe it geometrically (e.g., "intersection of a disk and a half-space").
c) Solve (geometrically[1]) the following semidefinite programs:
   i) $\min \; x-y \quad \text{s.t. } (x,y) \in S$
   ii) $\max \; x-y \quad \text{s.t. } (x,y) \in S$
d) The image of S under the linear map $f\colon \mathbb{R}^2 \to \mathbb{R}$, $f(x,y) = x-y$, is a spectrahedral shadow. It is a convex subset of $\mathbb{R}$, hence an interval. Compute the endpoints of this interval.

Solution to Exercise 1. See Mathematica file `lecture22.nb` for details.

a) Analyzing the inequalities given by leading minors of each matrix, we obtain the following:

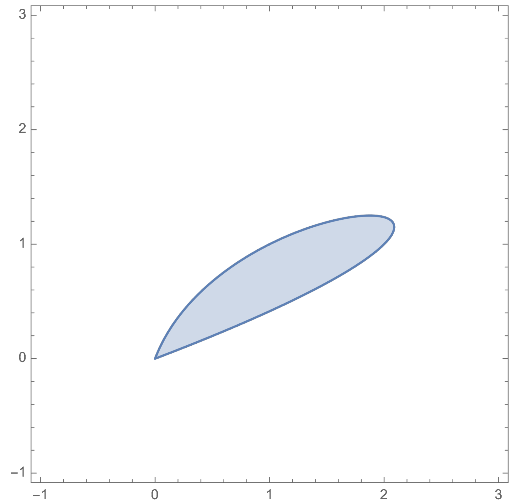


After some simplifications and using quantifier elimination to algorithmically check inclusions between semialgebraic sets, we find that the red region on the right (which corresponds to where the 3×3 matrix is positive-semidefinite) is entirely contained in the red region on the left (which corresponds to where the 2×2 matrix is positive-semidefinite). Thus, the spectrahedron S coincides with the red region on the right; in other words, we can write it as the basic closed semialgebraic set

$$S = \{(x,y) \in \mathbb{R}^2 : x \geq 0, \; y \geq 0, \; 3xy - x^2 - y^3 - y^2 \geq 0\}.$$

b) The region S is the intersection of the cubic $3xy - x^2 - y^3 - y^2 \geq 0$ with the positive quadrant $x \geq 0, \; y \geq 0$, and can be plotted as follows:

[1] i.e., analyzing the overlap of levelsets of the target function with the feasible set S.



c) Clearly, the extremal values of $f(x,y) := x - y$ are attained at the boundary of S, which is the portion of the cubic $g(x,y) := 3xy - x^2 - y^3 - y^2 = 0$ that lies in the first quadrant. In order to find these extremal points explicitly, we use the method of Lagrange multipliers. We compute $\nabla f(x,y) = (1,-1)$ and $\nabla g(x,y) = (-2x+3y, 3x-2y-3y^2)$. Thus, $\nabla f(x,y) = \lambda \nabla g(x,y)$ and $\lambda \neq 0$ is equivalent to $1/\lambda = -2x+3y = -(3x-2y-3y^2)$. Solving the polynomial system

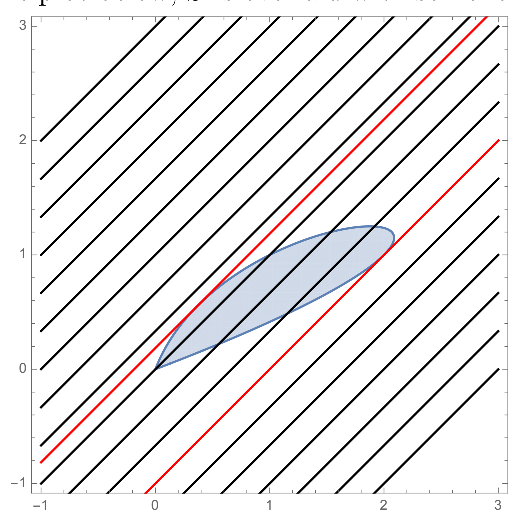
$$\begin{cases} -2x+3y = -(3x-2y-3y^2) \\ 3xy - x^2 - y^3 - y^2 = 0 \end{cases}$$

we find solutions $(0,0)$, $(\frac{10}{27}, \frac{5}{9})$, $(2,1)$, where the target function takes values $f(0,0) = 0$, $f(\frac{10}{27}, \frac{5}{9}) = -\frac{5}{27}$, $f(2,1) = 1$. Thus,

i) $\min_S x - y = -\frac{5}{27}$, attained at $(\frac{10}{27}, \frac{5}{9}) \in S$;

ii) $\max_S x - y = 1$, attained at $(2,1) \in S$.

Alternatively, one can solve $g(x,y) = 0$ locally as $x = x(y)$ and then substitute these solutions to obtain functions of a single variable $\phi(y) = f(x(y), y)$ whose minimum and maximum are the above extremal points. In the plot below, S is overlaid with some levelsets $\{(x,y) : f(x,y) = c\}$.



d) $f(S) = \left[-\frac{5}{27}, 1\right]$.