Lecture 18

1. Spectrahedra

Recall that a symmetric $n \times n$ matrix A is *positive-semidefinite*, which we denote by $A \succeq 0$, if for all $x \in \mathbb{R}^n$ we have $x^T A x \geq 0$. Similarly, A is called *positive-definite*, written $A \succ 0$, if for all $x \in \mathbb{R}^n \setminus \{0\}$, we have $x^T A x > 0$. We denote by $\mathcal{C}_{PSD} := \{X \in \mathrm{Sym}^2(\mathbb{R}^n) : X \succeq 0\}$ the (closed) convex cone of positive-semidefinite $n \times n$ matrices, whose interior is the (open) convex cone of positive-definite matrices.

The natural inner product in the vector space $\mathrm{Sym}^2(\mathbb{R}^n)$ of symmetric $n \times n$ matrices is given by $\langle X, Y \rangle = \operatorname{tr} XY$; so, given $A \in \mathrm{Sym}^2(\mathbb{R}^n)$ and $b \in \mathbb{R}$, the affine equation $\langle A, X \rangle = b$ determines a hyperplane in $\mathrm{Sym}^2(\mathbb{R}^n)$. A subset $S \subset \mathrm{Sym}^2(\mathbb{R}^n)$ is a *spectrahedron* if it is of the form

$$S = \big\{X \in \mathrm{Sym}^2(\mathbb{R}^n) : \langle A_i, X \rangle = b_i,\ 1 \leq i \leq m, \text{ and } X \succeq 0\big\}, \tag{1}$$

for some $A_i \in \mathrm{Sym}^2(\mathbb{R}^n)$, $1 \leq i \leq m$. Equivalently, S is a spectrahedron if it can be described by a *linear matrix inequality*, that is,

$$S = \big\{x \in \mathbb{R}^d : M_0 + x_1 M_1 + \cdots + x_d M_d \succeq 0\big\}, \tag{2}$$

where $M_j \in \mathrm{Sym}^2(\mathbb{R}^n)$, $0 \leq j \leq d$. Both definitions are equivalent, but before arguing that, we go over some examples.

Exercise 1. Use Sylvester's criterion to describe geometrically the following spectrahedra in $\mathbb{R}^2$:

a) $S = \left\{(x_1, x_2) \in \mathbb{R}^2 : \begin{pmatrix} 1 + x_1 & x_2 \\ x_2 & 1 - x_1 \end{pmatrix} \succeq 0\right\}$,

b) $S = \left\{(x_1, x_2) \in \mathbb{R}^2 : \begin{pmatrix} 1 + x_1 & & & \\ & 1 - x_1 & & \\ & & 1 + x_2 & \\ & & & 1 - x_2 \end{pmatrix} \succeq 0\right\}$,

c) $S = \left\{(x_1, x_2) \in \mathbb{R}^2 : \begin{pmatrix} 1 & x_1 & x_2 \\ x_1 & 1 & x_1 \\ x_2 & x_1 & 1 \end{pmatrix} \succeq 0\right\}$.

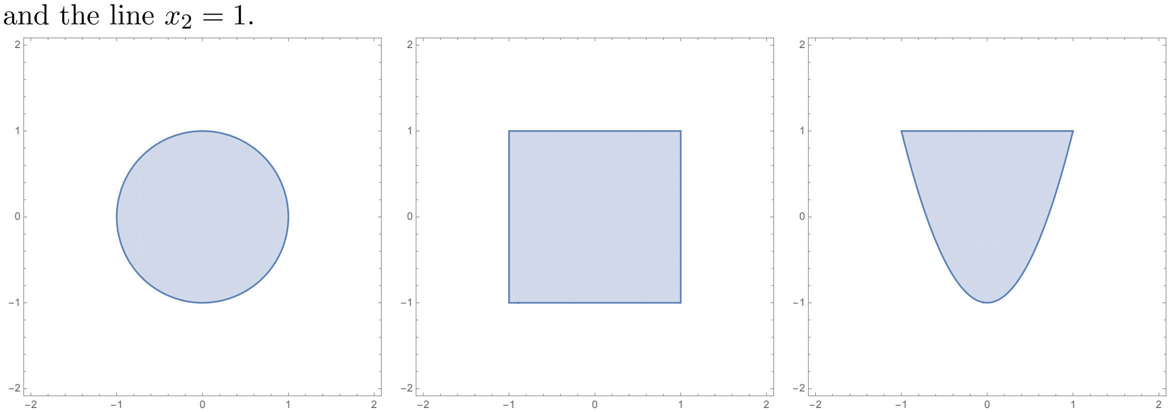
Note that the above are spectrahedral descriptions of the form (2). Can you find an equivalent description of the form (1) for each of them?

Solution to Exercise 1. The spectrahedra are as follows:

a) $S = \{(x_1, x_2) \in \mathbb{R}^2 : x_1^2 + x_2^2 \leq 1\}$ is the unit disk in $\mathbb{R}^2$.

b) $S = \{(x_1, x_2) \in \mathbb{R}^2 : -1 \leq x_1 \leq 1, -1 \leq x_2 \leq 1\}$ is a square in $\mathbb{R}^2$.

c) $S = \{(x_1, x_2) \in \mathbb{R}^2 : 2x_1^2 - 1 \leq x_2 \leq 1\}$ is the bounded region between the parabola $x_2 = 2x_1^2 - 1$ and the line $x_2 = 1$.



Exercise 2. Show that the spectrahedral descriptions (1) and (2) are equivalent.

Solution to Exercise 2. To show that a spectrahedron of the form (1) can be described in the form (2), note that $\{X \in \mathrm{Sym}^2(\mathbb{R}^n) : \langle A_i, X\rangle = b_i,\ 1 \le i \le m\}$ is an affine subspace of $\mathrm{Sym}^2(\mathbb{R}^n)$, hence it is a translation of a linear subspace $\mathrm{span}\{M_1, \dots, M_d\} \subset \mathrm{Sym}^2(\mathbb{R}^n)$ by a vector $M_0 \in \mathrm{Sym}^2(\mathbb{R}^n)$. Thus, $S = \{X \in \mathrm{Sym}^2(\mathbb{R}^n) : X = M_0 + x_1 M_1 + \cdots + x_d M_d \text{ for some } x \in \mathbb{R}^d, \text{ and } X \succeq 0\}$, so it is of the form (2).

Conversely, to show that a spectrahedron of the form (2) can be described in the form (1), let $\varphi(x) = M_0 + x_1 M_1 + \cdots + x_d M_d$, so that $S = \{x \in \mathbb{R}^d : \varphi(x) \succeq 0\}$. The map $\varphi\colon \mathbb{R}^d \to \mathrm{Sym}^2(\mathbb{R}^n)$ is affine-linear, hence its image is an affine subspace of $\mathrm{Sym}^2(\mathbb{R}^n)$. In particular, there exist some $A_i \in \mathrm{Sym}^2(\mathbb{R}^n)$ and $b_i \in \mathbb{R}$, $1 \le i \le m$, so that $\mathrm{im}\,\varphi = \{X \in \mathrm{Sym}^2(\mathbb{R}^n) : \langle A_i, X\rangle = b_i,\ 1 \le i \le m\}$. Thus, $S = \{X \in \mathrm{Sym}^2(\mathbb{R}^n) : X = \varphi(x) \text{ for some } x \in \mathbb{R}^d, \text{ and } X \succeq 0\}$ is of the form (1).

Exercise 3. Prove that spectrahedra are convex.

Solution to Exercise 3. Spectrahedra are intersections of affine subspaces with the convex cone of positive-semidefinite matrices, and the intersection of convex sets is convex.

Exercise 4. Prove that all polyhedra are spectrahedra. Clearly, the converse is not true, but find a *necessary and sufficient* condition on the matrices $M_j \in \mathrm{Sym}^2(\mathbb{R}^n)$ for (2) to be a polyhedron.

Solution to Exercise 4. A polyhedron is described by finitely many, say n, affine linear inequalities. Thus, it can be written as the set of $x \in \mathbb{R}^d$ such that $M_0+x_1 M_1+\cdots+x_d M_d \succeq 0$ for diagonal matrices $M_j \in \mathrm{Sym}^2(\mathbb{R}^n)$, and is hence of the form (2). A necessary and sufficient condition for $M_0 + x_1 M_1 + \cdots + x_d M_d \succeq 0$ to describe a polyhedron is that all matrices M_j commute, in which case they are simultaneously diagonalizable.