Lecture 16

1. Review of Linear Algebra

Recall that $\lambda \in \mathbb{R}$ is an *eigenvalue* for an $n \times n$ matrix A if the linear equation $Ax = \lambda x$ has a nontrivial solution $x \neq 0$. In other words, the kernel (or nullspace) of $\lambda \operatorname{Id} - A$ is nontrivial; so, equivalently, the eigenvalues of A are the roots of the *characteristic polynomial* $p(\lambda) = \det(\lambda \operatorname{Id} - A)$, which is a monic polynomial of degree n in λ.

If λ is an eigenvalue of A, the elements of $\ker(\lambda \operatorname{Id} - A)$ are called the *eigenvectors* of A associated to the eigenvalue λ, and the linear subspace $\ker(\lambda \operatorname{Id} - A)$ is called the *eigenspace* of A associated to the eigenvalue λ.

Exercise 1. Find all eigenvalues of the following matrices; for each eigenvalue, find a basis of the corresponding eigenspace.

a) $A = \begin{pmatrix} 3 & 4 \\ 4 & 3 \end{pmatrix}$

b) $A = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 2 & 4 \end{pmatrix}$

c) $A = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}$

The collection of all eigenvalues of the matrix A is called its *spectrum*, and we denote it by $\operatorname{Spec}(A) = \{\lambda_i \in \mathbb{R} : \lambda_i \text{ is an eigenvalue of } A\}$. The matrix A is *diagonalizable* if there is a decomposition $\mathbb{R}^n = \bigoplus_{\lambda_i \in \operatorname{Spec}(A)} \ker(\lambda_i \operatorname{Id} - A)$ into eigenspaces associated to eigenvalues of A. Equivalently, A is diagonalizable if there exists an invertible matrix P such that $A = PDP^{-1}$, where $D = \operatorname{diag}(\lambda_i)$ is a diagonal matrix (in this case, the columns of P are the coordinates of eigenvectors of A). Which of the matrices above are diagonalizable, and with which P?

Note that the behavior of A as a map $A\colon \mathbb{R}^n \to \mathbb{R}^n$ is geometrically clear in each eigenspace (it is simply multiplication by λ_i) so if A is diagonalizable, then we can find out what A does to any vector in $\mathbb{R}^n$ by decomposing it into the components in each eigenspace of A. In other words, if we "change basis" using P and work on a basis of eigenvectors of A, then A operates as the diagonal matrix $D = \operatorname{diag}(\lambda_i)$. This is illustrated in the diagram below:

$$\begin{array}{ccc} \mathbb{R}^n & \xrightarrow{\;A\;} & \mathbb{R}^n \\ {\scriptstyle P}\uparrow & & \uparrow{\scriptstyle P} \\ \mathbb{R}^n & \xrightarrow{\;D\;} & \mathbb{R}^n \end{array}$$

We will make frequent use of the following important algebraic results.

Theorem 1. *If A is symmetric, that is, $A^T = A$, then A is orthogonally diagonalizable, that is, there exists a matrix P such that $P^T P = \operatorname{Id}$ and $A = PDP^{-1} = PDP^T$, where D is diagonal.*

Theorem 2. *The characteristic polynomial $p(\lambda)$ of an $n \times n$ matrix A satisfies*

$$p(\lambda) = \det(\lambda \operatorname{Id} - A) = \sum_{k=0}^{n} (-1)^k \operatorname{tr}(\wedge^k A)\, \lambda^{n-k},$$

where $\operatorname{tr}(\wedge^k A)$ is the sum of all principal minors[1] of A of size k.

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[1] Recall that a principal minor of A of size k is the determinant of a submatrix of A obtained by selecting rows $\{i_1, \dots, i_k\}$ and columns $\{i_1, \dots, i_k\}$.