Lecture 14

1. Duality

1.1. **Estimating the optimal value.** Suppose we are given the following LP[1]

$$\begin{aligned} \max \quad 2x_1 + 3x_2 \quad \text{s.t.} \quad 4x_1 + 8x_2 &\leq 12 \\ 2x_1 + x_2 &\leq 3 \\ 3x_1 + 2x_2 &\leq 4 \\ x_1, x_2 &\geq 0. \end{aligned} \tag{1}$$

Solving it, e.g., via the simplex method, we find the optimal value is $\frac{19}{4}$, attained at $x_1 = \frac{1}{2}$, $x_2 = \frac{5}{4}$. However, suppose we did not have time (or were not interested enough) to solve this LP exactly, and just wanted an *upper bound* for the solution. For example, from the first constraint, we get

$$2x_1 + 3x_2 \leq 4x_1 + 8x_2 \leq 12,$$

and, indeed, $\frac{19}{4} < 12$. A better (smaller) upper bound is found dividing the first constraint by 2.

Exercise 1. Add the first two constraints and divide by 3 to obtain an even better upper bound for the optimal value of the above LP. How large can it be? Can you improve that bound further?

Solution to Exercise 1. Adding the first two constraints and dividing by 3 we find

$$2x_1 + 3x_2 = \tfrac{1}{3}(4x_1 + 8x_2) + \tfrac{1}{3}(2x_1 + x_2) \leq 5,$$

so we have that 5 is an upper bound. More generally, taking linear combinations of the 3 constraints, namely, multiplying them by y_1, y_2, y_3, respectively, and adding the results, we have

$$(4y_1 + 2y_2 + 3y_3)x_1 + (8y_1 + y_2 + 2y_3)x_2 \leq 12y_1 + 3y_2 + 4y_3.$$

Since all variables are nonnegative, the above is useful to find upper bounds for the optimal solution of the LP (1) if and only if the coefficients of x_1 and of x_2 are at least 2 and 3, respectively.

Thus, the optimal choice of y_1, y_2, y_3, i.e., the one that finds the smallest possible upper bound, is the solution to the LP

$$\begin{aligned} \min \quad 12y_1 + 3y_2 + 4y_3 \quad \text{s.t.} \quad 4y_1 + 2y_2 + 3y_3 &\geq 2, \\ 8y_1 + y_2 + 2y_3 &\geq 3, \\ y_1, y_2, y_3 &\geq 0. \end{aligned} \tag{2}$$

The optimal solution to this LP is $\frac{19}{4}$, attained at $y_1 = \frac{5}{16}, y_2 = 0, y_3 = \frac{1}{4}$. Does $\frac{19}{4}$ look familiar? The LP (2) is called the *dual* of the *primal* LP (1).

More generally, the *dual* LP to the *primal*

$$\max \quad c^T x \quad \text{s.t.} \quad Ax \leq b, \;\; x \geq 0, \tag{3}$$

is given by

$$\min \quad b^T y \quad \text{s.t.} \quad A^T y \geq c, \;\; y \geq 0. \tag{4}$$

Exercise 2. Find the dual LP to

$$\begin{aligned} \max \quad 34x_1 + 31x_2 \quad \text{s.t.} \quad 5x_1 + 2x_2 &\leq 16 \\ 3x_1 + 7x_2 &\leq 27 \\ x_1, x_2 &\geq 0. \end{aligned}$$

Solve both primal and dual LP.

[1]This example is from Sec. 6.1 in "Understanding and Using Linear Programming", by Jiri Matousek and Bernd Gärtner (Springer).

Solution to Exercise 2. The dual LP is

$$\begin{aligned} \min \quad 16y_1 + 27y_2 \quad \text{s.t.} \quad 5y_1 + 3y_2 &\geq 34 \\ 2y_1 + 7y_2 &\geq 31 \\ y_1, y_2 &\geq 0. \end{aligned}$$

The optimal solution to the primal LP is $x_1 = 2$, $x_2 = 3$, where the target function achieves its maximum 161. The optimal solution to the dual LP is $y_1 = 5$, $y_2 = 3$, where the target function achieves its minimum 161.

What would happen if we take a dual again?

Proposition 1. *The dual of the dual of a LP is the original LP itself.*

Exercise 3. Prove Proposition 1, that is, show that the dual of (4) is (3).

Solution to Exercise 3. The LP (4) can be equivalently stated as as

$$\max \quad -b^T y \quad \text{s.t.} \quad -A^T y \leq -c, \;\; y \geq 0.$$

Thus, its dual is

$$\min \quad -c^T x \quad \text{s.t.} \quad -(A^T)^T x \leq -b, \;\; x \geq 0,$$

which can be equivalently stated as (3).

More generally, using the routine tricks (multiplying both sides by -1 to exchange $\leq$ and $\geq$, writing an unconstrained variable as the difference of nonnegative variables, etc.), we can dualize any LP. Namely, if the primal LP has target function $c^T x$, $x \in \mathbb{R}^n$, constraints given by an $m \times n$ matrix A, and right-hand sides given by a vector $b \in \mathbb{R}^m$, then the dual can be obtained as follows:

Primal LP	Dual LP
$m = \#$ constraints	$m = \#$ variables (y_i)
$n = \#$ variables (x_j)	$n = \#$ constraints
max	min
$\leq b_i$	$y_i \geq 0$
$= b_i$	y_i unconstrained
$x_j \geq 0$	$\geq c_j$
x_j unconstrained	$= c_j$

Note that the above table can be read left to right, or right to left, as a consequence of Proposition 1.

Exercise 4. Find the dual LP to

$$\begin{aligned} \min \quad 3x_2 + x_3 \quad \text{s.t.} \quad x_1 + 3x_2 &\leq 10 \\ 2x_1 - x_2 + x_3 &\geq 5 \\ 5x_1 - 3x_2 + 4x_3 &= 15 \\ x_1 &\geq 0, \\ x_2, x_3 &\text{ unconstrained.} \end{aligned}$$

Solution to Exercise 4.

$$\begin{aligned} \max \quad -10y_1 + 5y_2 + 15y_3 \quad \text{s.t.} \quad -y_1 + 2y_2 + 5y_3 &\leq 0 \\ -3y_1 - y_2 - 3y_3 &= 3 \\ y_2 + 4y_3 &= 1 \\ y_1 \geq 0, y_2 &\geq 0 \\ y_3 &\text{ unconstrained.} \end{aligned}$$

Finally, let us address what we saw empirically in the beginning of the lecture:

Proposition 2 (Weak Duality)**.** *If x and y are feasible solutions to* (3) *and* (4)*, respectively, then* $c^T x \leq b^T y$.

Proof. Since $A^T y \geq c$, we have that $c^T \leq (A^T y)^T = y^T A$. Taking the inner product on both sides with $x \geq 0$, it follows that $c^T x \leq y^T Ax$. Similarly, taking the inner product of $Ax \leq b$ and $y \geq 0$, we find $y^T Ax \leq y^T b$. Concatenating these inequalities, we obtain $c^T x \leq y^T Ax \leq y^T b = b^T y$. $\square$

In fact, much more can be said about solving a pair of dual LPs:

Theorem 1 (Strong Duality)**.** *For the LPs* (3) *and* (4)*, exactly one of the following holds:*

(i) *Neither* (3) *nor* (4) *has a feasible solution;*

(ii) (3) *is unbounded and* (4) *has no feasible solution;*

(iii) (3) *has no feasible solution and* (4) *is unbounded;*

(iv) *Both* (3) *and* (4) *have feasible solutions, say x_* and y_*. In this case, the optimal values are equal, that is,* $c^T x_* = b^T y_*$.

A proof of the above result (using the simplex method!) can be found in the suggested textbooks.