## Independent Events

**Def:** $E$ and $F$ are <u>independent</u> events if
$$P(EF) = P(E)\cdot P(F).$$

**Note:** If $E$ and $F$ are indep.:
$$P(E|F) = \frac{P(EF)}{P(F)} = \frac{P(E)\cdot \cancel{P(F)}}{\cancel{P(F)}} = P(E)$$
$$P(F|E) = P(F).$$

**Examples:**

① A card is selected at random (52 cards in a deck)

$E$ = card is an Ace

$F$ = card is of hearts

$$P(E) = \frac{4}{52} = \frac{1}{13}, \quad P(F) = \frac{13}{52} = \frac{1}{4}, \quad P(\underbrace{EF}_{\text{The Ace of hearts}}) = \frac{1}{52}$$

$$P(E)\cdot P(F) = \frac{1}{13}\cdot\frac{1}{4} = \frac{1}{52} = P(EF)$$

$\Rightarrow$ $E$ and $F$ are <u>independent</u>.

② Toss 2 fair dice

$E$ = sum is 6

$F$ = first is 4

**Exercise:** What about $\tilde{E}$ = sum is 7

(Ans.: $\tilde{E}$ and $F$ are independent.)

$$P(E)=\frac{5}{36},\qquad P(F)=\frac{1}{6},\qquad P(EF)=\frac{1}{36}$$

$$\left.\begin{array}{l} 1+5=6\\ 2+4=6\\ 3+3=6\\ \boxed{4+2=6}\\ 5+1=6 \end{array}\right\} 5$$

$$P(E)\cdot P(F)=\frac{5}{36}\cdot\frac{1}{6}=\frac{5}{216}\neq\frac{1}{36}=P(EF)$$

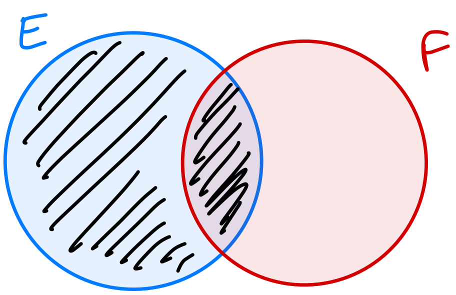
$\Longrightarrow$ $E$ and $F$ are not independent.

$$\underline{6}\;\underline{6}=36$$

Prop: If $E$ and $F$ are independent, then so are $E$ and $F^c$.

Pf: By def., $E$ and $F$ are indep. if
$$P(E)\cdot P(F)=P(EF).$$

Want to show:
$$P(E)\cdot P(F^c)=P(EF^c)$$




$$E=EF\cup EF^c$$
$$P(E)=P(EF)+P(EF^c)$$
$$=P(E)\cdot P(F)+P(EF^c)$$
$$\Longrightarrow\quad P(E)-P(E)\cdot P(F)=P(EF^c)$$
$$P(E)\big(\underbrace{1-P(F)}_{P(F^c)}\big)=P(EF^c)$$
$$P(E)\cdot P(F^c)=P(EF^c).$$

$\square$

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However: If $\begin{cases} E \text{ and } F \text{ are independent}, \\ E \text{ and } G \text{ are independent} \end{cases}$

is it true that $E$ and $FG$ are independent?

NO: Here is a counter-example

Toss 2 dice:

$E =$ sum is $7$

$F =$ first is $3$

$G =$ second is $4$

$E$ and $F$ are indep. ✓

$E$ and $G$ are indep. ✓

$E$ and $FG$ are not indep: ✗

$$P(E|FG) = 1 > P(E)$$

What does it mean for $E, F, G$ to be independent?

Def: $E, F, G$ are indep. if

$$\left.\begin{aligned} P(EFG) &= P(E)\cdot P(F)\cdot P(G) \\ P(EF) &= P(E)\cdot P(F) \\ P(EG) &= P(E)\cdot P(G) \\ P(FG) &= P(F)\cdot P(G). \end{aligned}\right\}$$

need to have all of these.

What about for collections of more than 3 events?

Def: $\{E_i\}$ are independent if for all subsets

$$\{E_{i_1}, E_{i_2}, \ldots E_{i_r}\}, \qquad P(E_{i_1}E_{i_2}\cdots E_{i_r}) = P(E_{i_1})\cdot P(E_{i_2})\cdots P(E_{i_r}).$$

for any $r$

Def: $\{E_i\}$ are called trials if $\{E_i\}$ are independent, and have the same possible outcomes

"$E_i =$ outcome of $i^{th}$ repetition of a given experiment".

# Bernoulli process

Suppose an infinite sequence of (independent) trials is conducted, and the outcome is either a success (S) or a failure (F).

SSFSSSFFSFFSS....

$$\begin{cases} \text{Prob. of Success} = p \\ \text{Prob. of Failure} = 1-p. \end{cases}$$

a) What is the probability of <u>at least</u> 1 success in the first n trials?

$$P\begin{pmatrix} \text{At least} \\ 1 \ S \end{pmatrix} = 1 - P\begin{pmatrix} \text{No } S \\ \text{at all} \end{pmatrix} = \boxed{1-(1-p)^n}$$

b) What is the probability of <u>exactly</u> K successes in the first n trials?

$$\underbrace{SSS\ldots S}_{K}\ \underbrace{FF\ldots F}_{n-K}$$

$$\boxed{\binom{n}{K} p^K \cdot (1-p)^{n-K}}$$

c) What is the probability that the first n trials are a success? What happens as $n \nearrow +\infty$?

$$\underbrace{S\ldots S}_{n} \qquad \lim_{n\nearrow+\infty} p^n = \begin{cases} 0, & 0 \le p < 1 \\ 1, & p = 1 \end{cases}$$

$$\underbrace{F\ldots F}_{n} \qquad \lim_{n\nearrow+\infty} (1-p)^n = \begin{cases} 0, & 0 \le 1-p < 1 \ (1 \ge p > 0) \\ 1, & 1-p = 1 \ (p=0) \end{cases}$$

Ex: Rolling a pair of dice infinitely many times. What is the prob. that a sum of 5 occurs before a sum of 7?

$$E_n = \begin{pmatrix} \text{no 5 or 7 occurs in the first } n-1 \text{ trials} \\ \text{and 5 occurs precisely on the } n^{th} \text{ trial.} \end{pmatrix}$$

"you have not won/loss until $n^{th}$ trial, and you win on the $n^{th}$ trial"

$$\text{Desired probability} = P\Big(\underbrace{\bigcup_{n=1}^{+\infty} E_n}_{\substack{\text{at least} \\ \text{1 of them} \\ \text{must happen}}}\Big) \overset{\text{disjoint}}{=} \sum_{n=1}^{+\infty} P(E_n) = \sum_{n=1}^{+\infty} \frac{4}{36}\left(1-\frac{10}{36}\right)^{n-1} = ?$$

"prob. that you win"

$P(5) = 4/36$

$P(7) = 6/36$

$$P(5^c \cap 7^c) = P\left((5 \cup 7)^c\right) = 1 - P(5 \cup 7)$$
$$= 1 - (P(5) + P(7))$$
$$= 1 - \frac{4+6}{36} = 1 - \frac{10}{36}.$$

$$P(E_n) = \underbrace{\frac{4}{36}}_{\substack{\text{5 occurs} \\ \text{on the } n^{th} \\ \text{trial}}} \cdot \underbrace{\left(1 - \frac{10}{36}\right)^{n-1}}_{\substack{\text{no 5 nor 7} \\ \text{on trials} \\ 1,\dots,n-1}}$$

$$\sum_{n=1}^{+\infty} \frac{4}{36}\left(1-\frac{10}{36}\right)^{n-1} = \frac{1}{9} \sum_{n=1}^{+\infty} \left(\frac{13}{18}\right)^{n-1} = \frac{1}{9} \frac{1}{1-\left(\frac{13}{18}\right)}$$

$$= \frac{1}{9} \frac{18}{5} = \boxed{\frac{2}{5}} \qquad 40\%$$

Note: $P(5) = \frac{4}{36}$, $P(7) = \frac{6}{36}$

$\Rightarrow$ Odds are $6:4$ $\Rightarrow P = \frac{4}{4+6} = \frac{4}{10} = 40\%$.

In general, if $E$ and $F$ are mutually exclusive (i.e. disjoint), performing (indep.) trials, the probability that $E$ occurs before $F$ is

$$\frac{P(E)}{P(E)+P(F)}.$$

Ex: The probability of some category 5 hurricane hitting NYC any given year is 10%. (assume this prob is constant, despite the fact that, due to climate change, it is actually increasing with time!)

Bernoulli process

S = cat 5 hurricane occurs in given yr $\quad p = 1/10$

F = - - - does not occur in given yr $\quad 1-p = 9/10$

a) What is the prob. that no cat. 5 hurricanes hit NYC for 15 straight years?

$$(1-p)^{15} = \left(\frac{9}{10}\right)^{15} \cong \underline{20.59\%}$$

b) What is the prob. that cat. 5 hurricanes hit NYC in exactly 3 years $\underbrace{\text{between 2021 and 2031}}_{\text{10 years.}}$?

$$P\left(\begin{matrix}3 \text{ s.} \\ \text{in } 10 \text{ yrs}\end{matrix}\right) = \binom{10}{3} \cdot p^3 (1-p)^7 = \binom{10}{3} \frac{1}{10^3} \frac{9^7}{10^7} = \binom{10}{3} \frac{9^7}{10^{10}} \cong \underline{5.74\%}$$

c) What is the prob. that cat. 5 hurricanes hit NYC in at most 2 years $\underbrace{\text{between 2021 and 2031}}_{\text{10 years.}}$?

Cat. 5 hurricanes in

$$0 \text{ yr} \longrightarrow (1-p)^{10} = \left(\frac{9}{10}\right)^{10} = \frac{9^{10}}{10^{10}}$$

$$1 \text{ yr} \longrightarrow \binom{10}{1} p (1-p)^9 = 10 \cdot \frac{1}{10} \cdot \left(\frac{9}{10}\right)^9 = \frac{9^9}{10^9}$$

$$2 \text{ yrs} \longrightarrow \binom{10}{2} p^2 (1-p)^8 = \binom{10}{2} \frac{1}{10^2} \frac{9^8}{10^8} = \binom{10}{2} \frac{9^8}{10^{10}}$$

$$P = \frac{9^{10}}{10^{10}} + \frac{9^{9}}{10^{9}} + \binom{10}{2}\frac{9^{8}}{10^{10}} \simeq \boxed{92.98\%}$$