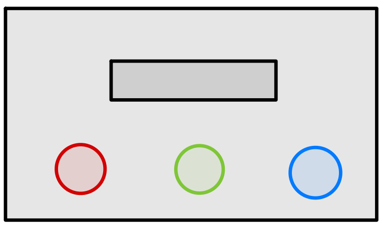
Conditional Probability



R = red light blinks
G = green light blinks
B = blue light blinks

disjoint

$$P(R) = \frac{1}{5} \qquad P(G) = \frac{2}{5} \qquad \cancel{P(B) = \frac{2}{5}}$$

suppose the blue light breaks.

- what are the "new" probabilities $\tilde{P}$ for the events R and G, given that B does not happen?

$$\begin{cases} \tilde{P}(R) + \tilde{P}(G) = 1 \\ \tilde{P}(G) = 2 \cdot \tilde{P}(R) \end{cases} \qquad \tilde{P}(R) = \frac{1}{3} \qquad \tilde{P}(G) = \frac{2}{3}$$

- Let us analyze this "distributing" the old probability of B happening to new probabilities.

Information: B does not happen

$$\boxed{B^c \text{ happens.}}$$

$$P(B^c) = 1 - P(B) = 1 - \frac{2}{5} = \frac{3}{5}.$$

$$\boxed{\tilde{P}(A) = \frac{P(A \cap B^c)}{P(B^c)}}$$

new prob.

new information

Since original events are disjoint:

$$P(R \cap B^c) = P(R)$$

$$P(G \cap B^c) = P(G)$$

$$\tilde{P}(R) = \frac{P(R \cap B^c)}{P(B^c)} = \frac{P(R)}{P(B^c)} = \frac{1/5}{3/5} = \boxed{\frac{1}{3}}$$

$$\tilde{P}(G) = \frac{P(G \cap B^c)}{P(G)} = \frac{P(G)}{P(B^c)} = \frac{2/5}{3/5} = \boxed{\frac{2}{3}}$$

Def: The conditional probability that an event E happens given that the event F happens is

$$P(E|F) = \frac{P(EF)}{P(F)} \qquad (\text{assuming } P(F) > 0).$$

"given that"

Compare w/ example above: $\tilde{P}(E) = P(E|B^c)$

Facts about conditional probabilities

(1) If $P: \mathcal{P}(\Omega) \to [0,1]$ is a probability and $F \in \mathcal{P}(\Omega)$ is such that $P(F) > 0$, then $\tilde{P}: \mathcal{P}(\Omega) \to [0,1]$ given by

$$\tilde{P}(E) := P(E|F) = \frac{P(EF)}{P(F)}$$

is a probability on Ω:

- $\tilde{P}(E) = P(E|F) \in [0,1]$
- $\tilde{P}(\Omega) = P(\Omega|F) = 1$
- If E_i are pairwise disjoint, i.e., $E_i \cap E_j = \emptyset$ if $i \neq j$, then $\tilde{P}\left(\bigcup_{i=1}^{n} E_i\right) = P\left(\bigcup_{i=1}^{n} E_i \middle| F\right) = \sum_{i=1}^{n} \underbrace{P(E_i|F)}_{\tilde{P}(E_i)}$

So all previous results apply to cond. prob.: for example, $P(E^c|F) = 1 - P(E|F)$

(2) In particular, if Ω is finite and all outcomes are equally likely, then

$$P(E|F) = \frac{P(EF)}{P(F)} = \frac{|E \cap F|}{|F|}$$

Proof: $P(EF) = \frac{|E\cap F|}{|\Omega|}$ $P(F) = \frac{|F|}{|\Omega|}$

$$P(E|F) = \frac{P(EF)}{P(F)} = \frac{|E\cap F|/|\Omega|}{|F|/|\Omega|} = \frac{|E\cap F|}{|F|} \qquad \square$$

(here $\Omega = S$ is the sample space)

Example: Two fair coins are tossed. What is the prob. of both landing on heads given that

a) the first coin lands on heads.

b) at least one coin lands on heads.

$\Omega = \{(T,T), (T,H), (H,T), (H,H)\}$ sample space.

a) $E = \{(H,H)\}$

$F = \{(H,T), (H,H)\}$ "first coin lands on heads"

$$P(E|F) = \frac{P(EF)}{P(F)} = \frac{|E\cap F|}{|F|} = \frac{1}{2}$$

$E\cap F = E$

b) $E = \{(H,H)\}$

$G = \{(H,T), (T,H), (H,H)\}$. "at least one heads"

$E \cap G = E$

$$P(E|G) = \frac{P(EG)}{P(G)} = \frac{|E \cap G|}{|G|} = \frac{1}{3}$$

Prop: (Multiplication Rule). Suppose $E_1, \dots, E_n$ are events s.t. $P(E_1 \cdots E_k) > 0$ for all $1 \le k \le n$. Then:

$$P(E_1 E_2 \dots E_n) = P(E_1) \cdot P(E_2|E_1) \cdot P(E_3|E_1 E_2) \dots P(E_n|E_1 \dots E_{n-1})$$

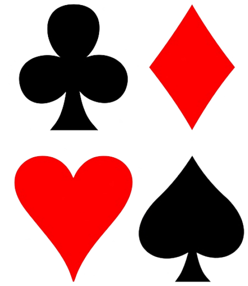
Pf: Computing the right-hand side of the above:

$$\cancel{P(E_1)} \frac{P(\cancel{E_1 E_2})}{P(\cancel{E_1})} \frac{P(\cancel{E_1 E_2 E_3})}{P(\cancel{E_1 E_2})} \cdots \frac{P(E_1 \dots E_{n-1} E_n)}{P(\cancel{E_1 \dots E_{n-1}})}$$

$= P(E_1 \cdots E_n)$.

(Can be made rigorous using induction)

Ex: A standard deck with 52 cards is randomly divided into 4 piles (with 13 cards each). What is the prob. that each pile contains exactly 1 ace?

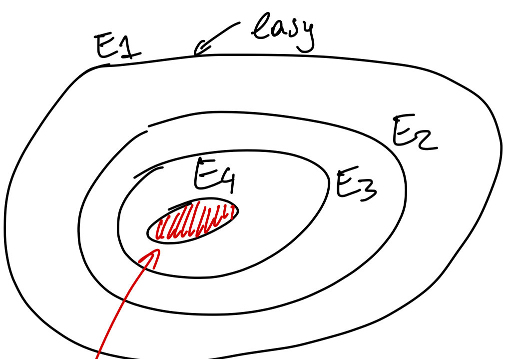


$E_1 = \{A_{\heartsuit}$ goes to some pile$\}$

$E_2 = \{A_{\spadesuit}$ and $A_{\heartsuit}$ go to different piles$\}$

$E_3 = \{A_{\spadesuit}, A_{\diamondsuit}$, and $A_{\heartsuit}$ go to different piles$\}$.

$E_4 = \{$All A's go in different piles$\}$



Question: $P(E_4) = ?$

$$P(E_1) = 1$$

want Nesting: $P(E_4) = P(E_1 E_2 E_3 E_4)$

Mult. rule

$$= P(E_1) \cdot P(E_2|E_1) \cdot P(E_3|E_1E_2) \cdot P(E_4|E_1E_2E_3).$$

$A_{\spadesuit}$ goes to the same pile as $A_{\heartsuit}$

$$P(E_2|E_1) = 1 - P(E_2^c|E_1) = 1 - \frac{12}{51}$$

$$P(E_3|E_1E_2) = 1 - P(E_3^c|E_1E_2) = 1 - \frac{24}{50}$$

$$P(E_4|E_1E_2E_3) = 1 - P(E_4^c|E_1E_2E_3) = 1 - \frac{36}{49}$$

$$P(E_4) = P(E_1E_2E_3E_4) = 1 \cdot \left(1 - \frac{12}{51}\right) \cdot \left(1 - \frac{24}{50}\right) \cdot \left(1 - \frac{36}{49}\right)$$

$$= \frac{2197}{20825} \cong 10.55\%$$

Alternative solution: Distribute 4 aces to the 4 piles that have 13 cards each.

$$P = \frac{\binom{13}{1}\binom{13}{1}\binom{13}{1}\binom{13}{1}}{\binom{52}{4}} = \frac{13^4}{\binom{52}{4}} = \frac{2197}{20825}$$

Ex: You lost your keys and are 80% sure they are in one of 2 pockets in your coat, being 40% sure they are on the Left pocket, and 40% sure they are on the Right pocket. If you look in the Left pocket and they are not there, what is the prob. you find them in the Right?

L = Keys are in left pocket

R = Keys are in right pocket.

$$P(L) = P(R) = \frac{40}{100} = \frac{2}{5} \qquad P(R|L^c) = ?$$

$$P(R|L^c) = \frac{P(R \cap L^c)}{P(L^c)} = \frac{P(R)}{1-P(L)} = \frac{2/5}{1-2/5} = \boxed{\frac{2}{3}}$$

$R \cap L^c = R$ b/c $R \cap L = \emptyset$.