Moment Generating Functions:

Def: The moment generating function of a random variable X is the function $M: I \to \mathbb{R}$ given by $M(t) = E(e^{tX})$.

interval around zero.

In other words, if X is discrete:

$$M(t) = E(e^{tX}) = \sum_x e^{tx} \cdot p(x)$$

$p(x) = P(X=x)$ is the prob. mass function of X.

instead, if X is continuous:

$$M(t) = E(e^{tX}) = \int_{-\infty}^{+\infty} e^{tx} \cdot f(x)\, dx$$

$f(x)$ is the p.d.f. of X

Recall: Moments of a random variable X are:

1st moment: $E(X) = M'(0)$

2nd moment: $E(X^2) = M''(0)$

3rd moment: $E(X^3) = M'''(0)$

$\vdots$

nth moment: $E(X^n) = M^{(n)}(0)$

$$M(t) = E(e^{tX})$$

$$M(0) = E(e^{0.X}) = E(1) = 1.$$

- First derivative of Moment generating function at $t=0$:

$$M'(t) = \frac{d}{dt}E(e^{tX}) = E\left(\frac{d}{dt}e^{tX}\right) = E\left(e^{tX}.X\right)$$

$$M'(0) = E(e^{0.X}.X) = E(X).$$

- Second derivative of ——————————"——————————————"——————:

$$M''(t) = \frac{d}{dt}M'(t) = \frac{d}{dt}E(e^{tX}.X) = E\left(\frac{d}{dt}e^{tX}.X\right) = E\left(e^{tX}.X^2\right)$$

$$M''(0) = E(e^{0.X}.X^2) = E(X^2).$$

- Similarly, taking n derivatives of $M(t)$ gives:

$$M^{(n)}(t) = E\left(e^{tX}.X^n\right)$$

$$M^{(n)}(0) = E(e^{0.X}.X^n) = E(X^n). \longleftarrow$$ n^{th} moment of X.

<u>Remark</u>: This means that the Taylor Series of $M(t)$ at $t=0$ is:

$$\sum_{n=0}^{+\infty} \frac{M^{(n)}(0)}{n!}t^n = 1 + E(X)\,t + \frac{E(X^2)}{2!}t^2 + \frac{E(X^3)}{3!}t^3 + \dots + \frac{E(X^n)}{n!}t^n + \dots$$

Examples:

$X \sim \text{Binomial}(n, p)$

Recall:
$E(X) = n \cdot p$
$Var(X) = n \cdot p(1-p)$.

$$p(x) = \binom{n}{x} p^x (1-p)^{n-x}$$

$$M(t) = E(e^{tX}) = \sum_{x=0}^{n} e^{tx} p(x) = \sum_{x=0}^{n} e^{tx} \binom{n}{x} p^x (1-p)^{n-x}$$

$$= \sum_{x=0}^{n} \binom{n}{x} (\underbrace{e^t p}_{a})^x (\underbrace{1-p}_{b})^{n-x} = (\underbrace{e^t p}_{a} + \underbrace{1-p}_{b})^n$$

$$\boxed{M(t) = (e^t p + 1 - p)^n}$$

$$M'(t) = n(e^t p + 1 - p)^{n-1} \cdot (e^t p) \implies M'(0) = np = E(X)$$

$$M''(t) = n(n-1)(e^t p + 1 - p)^{n-2} (e^t p)^2 + n(e^t p + 1 - p)^{n-1} \cdot (e^t p)$$

$$M''(0) = n(n-1)p^2 + np = np((n-1)p + 1) = E(X^2)$$

E.g., $Var(X) = E(X^2) - E(X)^2$

$$= M''(0) - (M'(0))^2$$

$$= np((n-1)p + 1) - (np)^2$$

$$= np\Big(\underbrace{(n-1)p + 1 - np}_{\cancel{np} - p + 1 - \cancel{np}}\Big) = np(1-p)$$

$X \sim \text{Exponential}(\lambda)$

$$f(x) = \begin{cases} \lambda e^{-\lambda x} & \text{if } x \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

Recall:

$E(X) = \frac{1}{\lambda}$

$Var(X) = \frac{1}{\lambda^2}$

$$M(t) = E(e^{tX}) = \int_{-\infty}^{+\infty} e^{tx} f(x)\,dx = \int_{0}^{+\infty} e^{tx} \cdot \lambda e^{-\lambda x}\,dx$$

$$= \lambda \int_0^{+\infty} e^{-(\lambda - t)x}\,dx = \lambda \cdot \lim_{b \to \infty} \int_0^b e^{-(\lambda-t)x}\,dx$$

$$= \lambda \lim_{b\to\infty} \left(\left. \frac{e^{-(\lambda-t)x}}{-(\lambda-t)} \right|_0^b \right) = \lambda \lim_{b\to\infty} \left(\underbrace{\frac{e^{-(\lambda-t)b}}{-(\lambda-t)}}_{0 \text{ if } \lambda - t > 0} + \frac{1}{(\lambda-t)} \right)$$

$\lambda - t > 0$

$$\stackrel{\downarrow}{=} \frac{\lambda}{\lambda - t}$$

$$\boxed{M(t) = \frac{\lambda}{\lambda - t} \quad \text{if} \quad t < \lambda}$$

$$M'(t) = \frac{d}{dt}\left(\frac{\lambda}{\lambda - t}\right) = \frac{\lambda}{(\lambda - t)^2} \Rightarrow M'(0) = \frac{\lambda}{\lambda^2} = \frac{1}{\lambda} = E(X)$$

$$M''(t) = \frac{d}{dt}\left(\frac{\lambda}{(\lambda-t)^2}\right) = \frac{2\lambda}{(\lambda-t)^3} \Rightarrow M''(0) = \frac{2\lambda}{\lambda^3} = \frac{2}{\lambda^2} = E(X^2)$$

$$Var(X) = E(X^2) - E(X)^2 = M''(0) - (M'(0))^2 = \frac{2}{\lambda^2} - \left(\frac{1}{\lambda}\right)^2 = \frac{1}{\lambda^2}$$

$$X \sim \text{Normal}(\mu, \sigma^2)$$

$$f_X(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\frac{(x-\mu)^2}{\sigma^2}}$$

Recall:
$E(X) = \mu$
$Var(X) = \sigma^2$

First, consider a <u>standard</u> normal:

$$Z \sim \text{Normal}(0, 1)$$

$$f_Z(z) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}z^2}$$

$$M_Z(t) = E(e^{tZ}) = \int_{-\infty}^{+\infty} e^{tz} \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}} dz = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} e^{-\frac{z^2}{2}+tz} dz$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} e^{-\frac{(z-t)^2}{2}+\frac{t^2}{2}} dz = \frac{e^{t^2/2}}{\sqrt{2\pi}} \underbrace{\int_{-\infty}^{+\infty} e^{-\frac{(z-t)^2}{2}} dz}_{= \int_{-\infty}^{+\infty} e^{-s^2/2} ds = \sqrt{2\pi}}$$

$\begin{cases} s = z-t \\ ds = dz \end{cases}$

$$= e^{t^2/2}.$$

<u>Upshot</u>. $M_Z(t) = e^{t^2/2} = E(e^{tZ})$

Back to X: $\quad X = \mu + \sigma Z \quad$ (standardization).

$$M_X(t) = E(e^{tX}) = E(e^{t\mu + t\sigma Z}) = E(e^{t\mu} \cdot e^{t\sigma Z})$$

$$= e^{t\mu} \underbrace{E(e^{t\sigma Z})}_{= M_Z(t\sigma)} = e^{t\mu} M_Z(t\sigma) = e^{t\mu} e^{t^2\sigma^2/2} = e^{t\mu + \frac{t^2\sigma^2}{2}}$$

$$\boxed{\text{Upshot: } M_X(t) = e^{t\mu + \frac{t^2\sigma^2}{2}}}$$

The magic of Moment Generating Functions

Theorem. Suppose that $M_X(t) = E(e^{tX})$ is well-defined in an open interval around $x=0$. Then the distribution of X is uniquely determined by the function $M_X(t)$.

E.g. Suppose X has $M_X(t) = \dfrac{1}{1-t}$ if $t < 1$

Then: $X \sim \text{Exponential}(1)$

E.g. Suppose Y has $M_Y(t) = \left(\frac{1}{2}\right)^{10}(e^t + 1)^{10} = \left(\frac{e^t}{2} + \frac{1}{2}\right)^{10}$

Then: $Y \sim \text{Binomial}(10, \frac{1}{2})$. $= \left(\frac{1}{2}e^t + 1 - \frac{1}{2}\right)^{10}$

Another way to use this:

If X, Y are independent random variables,

$$M_{X+Y}(t) = E\left(e^{t(X+Y)}\right) = E\left(e^{tX} \cdot e^{tY}\right) \overset{\text{indep.}}{=} \underbrace{E(e^{tX})}_{M_X(t)}\underbrace{E(e^{tY})}_{M_Y(t)}$$

$$\boxed{M_{X+Y}(t) = M_X(t) \cdot M_Y(t)}$$

So, e.g., if $X_1 \sim \text{Normal}(\mu_1, \sigma_1^2)$ are indep, then:
$X_2 \sim \text{Normal}(\mu_2, \sigma_2^2)$

$$M_{X_1+X_2}(t) = M_{X_1}(t) \cdot M_{X_2}(t) = \left(e^{t\mu_1 + \frac{t^2\sigma_1^2}{2}}\right)\left(e^{t\mu_2 + \frac{t^2\sigma_2^2}{2}}\right)$$

$$= e^{t\mu_1 + \frac{t^2\sigma_1^2}{2} + t\mu_2 + \frac{t^2\sigma_2^2}{2}}$$

$$= e^{t(\mu_1+\mu_2) + \frac{t^2}{2}(\sigma_1^2+\sigma_2^2)} = M_{X_3}(t)$$

where $X_3 \sim \text{Normal}(\mu_1+\mu_2, \sigma_1^2+\sigma_2^2)$

By the above thm, it follows that

$$X_1 + X_2 \sim \text{Normal}(\mu_1+\mu_2, \sigma_1^2+\sigma_2^2).$$

Conditional expectation:

Discrete random variables:

Probability mass function of X given that $Y=y$:

$$p_{X|Y}(x,y) = P\left(X=x \mid Y=y\right) = \frac{P(X=x, Y=y)}{P(Y=y)} = \frac{p_{X,Y}(x,y)}{p_Y(y)}$$

Continuous random variables:

Probability density function of X given that $Y=y$:

$$f_{X|Y}(x,y) = \frac{f_{X,Y}(x,y)}{f_Y(y)} \qquad \text{if} \quad f_Y(y) > 0$$

Conditional expectation:

Discrete r.v.:

$$E(X \mid Y=y) = \sum_x x \underbrace{P(X=x \mid Y=y)}_{p_{X|Y}(x,y)} = \sum_x x\, p_{X|Y}(x,y)$$

Cont. r.v.:

$$E(X \mid Y=y) = \int_{-\infty}^{+\infty} x\, f_{X|Y}(x,y)\, dx$$

Conditional Expectation as a Random Variable:

$g(y) = E(X|Y=y)$ is a function of y

This defines a random variable $g(Y)$; denoted

$E(X|Y)$. ← is itself a random variable, that takes value $E(X|Y=y)$ when Y takes value y

Q: $E(E(X|Y)) = ?$

A: $\boxed{\text{Conditional Expectation Formula} \quad E(E(X|Y)) = E(X).}$ (holds for both discrete or cont. r.v.)

Pf: Assuming X, Y discrete;

$$E(E(X|Y)) = \sum_y E(X|Y=y)\, p_Y(y)$$

$$= \sum_y \sum_x x\, p_{X|Y}(x,y)\, p_Y(y)$$

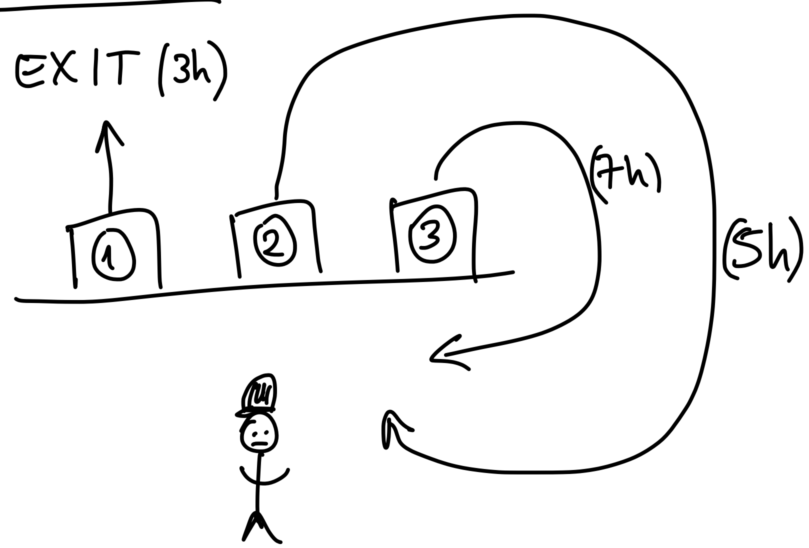
$$= \sum_y \sum_x x \, \frac{p_{X,Y}(x,y)}{\cancel{p_Y(y)}} \cdot \cancel{p_Y(y)}$$

$$= \sum_x x \Big(\underbrace{\sum_y p_{X,Y}(x,y)}_{p_X(x)} \Big) = \sum_x x \, p_X(x)$$

$$= E(X). \qquad \square$$

(Compare with:
$P(E) = P(E|F)P(F) + P(E|F^c)P(F^c)$)

Exercise: A miner is trapped in a mine with 3 doors.



Q: What is the expected number of hours that miner will take to reach exit?

X = # hours until exit

Y = door chosen.

Want:

$$E(X) = E(X|Y=1)P(Y=1) + E(X|Y=2)P(Y=2) + E(X|Y=3)P(Y=3)$$

$$= \frac{1}{3}\Big(\underline{E(X \mid Y=1)} + \underline{E(X \mid Y=2)} + \underline{E(X \mid Y=3)}\Big)$$

$$\left.\begin{aligned} E(X \mid Y=1) &= 3 \\ E(X \mid Y=2) &= 5 + E(X) \\ E(X \mid Y=3) &= 7 + E(X) \end{aligned}\right\}$$

$$= \frac{1}{3}\Big(\underline{3} + \underline{5 + E(X)} + \underline{7 + E(X)}\Big)$$

$$= \frac{1}{3}\big(15 + 2E(X)\big)$$

Thus: $E(X) = 5 + \frac{2}{3}E(X).$

$$\frac{1}{3}E(X) = 5 \implies \boxed{E(X) = 15}$$