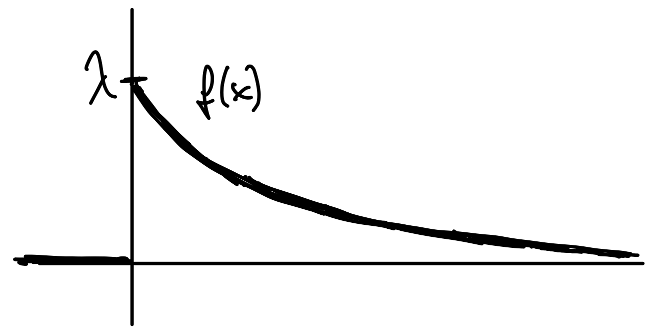
MAT330/681 Lecture 15 3/24/2021

Exponential Random Variable

A continuous random variable X is exponentially distributed if its p.d.f. is

$$f(x)=\begin{cases}\lambda e^{-\lambda x}, & x \geq 0\\ 0, & x<0\end{cases}$$



where $\lambda > 0$.

$$\int_{-\infty}^{+\infty} f(x)\,dx = \int_0^{+\infty} \lambda e^{-\lambda x}\,dx = \lambda \lim_{b\to\infty}\int_0^b e^{-\lambda x}\,dx =$$

$$= \lambda \lim_{b\to\infty}\left(\frac{e^{-\lambda x}}{-\lambda}\right)\Bigg|_0^b = \frac{\lambda}{-\lambda}\underbrace{\lim_{b\to\infty}\left(e^{-\lambda b}-1\right)}_{-1} = 1.$$

Expected Value:

$$E(X) = \int_{-\infty}^{+\infty} x f(x)\,dx = \int_0^{+\infty} x\lambda e^{-\lambda x}\,dx = \lambda \lim_{b\to\infty}\int_0^b x e^{-\lambda x}\,dx$$

Int. by Parts:

$$\int x e^{-\lambda x}\,dx = \frac{x e^{-\lambda x}}{-\lambda} - \int \frac{e^{-\lambda x}}{-\lambda}\,dx = \boxed{-\frac{x e^{-\lambda x}}{\lambda} - \frac{e^{-\lambda x}}{\lambda^2}}$$

$$u = x \qquad v = \frac{e^{-\lambda x}}{-\lambda}$$
$$du = dx \qquad dv = e^{-\lambda x}\,dx$$

$$\int u\,dv = uv - \int v\,du$$

$$= \cancel{\lambda}\cdot \lim_{b\to\infty}\left(-\frac{x e^{-\lambda x}}{\cancel{\lambda}} - \frac{e^{-\lambda x}}{\lambda^{\cancel{2}}}\right)\Bigg|_0^b = \lim_{b\to\infty}\left(\underbrace{-be^{-\lambda b}}_{\to 0\ (*)} - \underbrace{\frac{e^{-\lambda b}}{\lambda}}_{\to 0} + \frac{1}{\lambda}\right)$$

$$= \boxed{\frac{1}{\lambda}}$$

* L'Hospital: $\lim\limits_{b\to\infty} -be^{-\lambda b} = \lim\limits_{b\to\infty} -\frac{b}{e^{\lambda b}} \overset{L.H.}{=} \lim\limits_{b\to\infty} -\frac{1}{\lambda e^{\lambda b}} = 0.$

Variance

$$Var(X) = E(X^2) - E(X)^2$$

$$E(X^2) = \int_0^{+\infty} x^2 \lambda e^{-\lambda x}\,dx = \lim_{b\to\infty} \underbrace{\int_0^b \lambda x^2 e^{-\lambda x}\,dx}_{} =$$

$$\int \lambda x^2 e^{-\lambda x}\,dx \overset{\text{Parts}}{=} -e^{-\lambda x}\left(x^2 + \frac{2}{\lambda}x + \frac{2}{\lambda^2}\right)$$

$$= \lim_{b\to\infty} -e^{\lambda x}\left(x^2 + \frac{2}{\lambda}x + \frac{2}{\lambda^2}\right)\Big|_0^b$$

$$= \lim_{b\to\infty} \underbrace{\underset{\downarrow\,0}{-e^{-\lambda b}}\left(\underset{\downarrow\,+\infty}{b^2} + \underset{\downarrow\,+\infty}{\frac{2}{\lambda}b} + \frac{2}{\lambda^2}\right)}_{=0 \ \text{(L'Hospital)}} + \frac{2}{\lambda^2} = \frac{2}{\lambda^2}$$

$$Var(X) = \frac{2}{\lambda^2} - \left(\frac{1}{\lambda}\right)^2 = \frac{2-1}{\lambda^2} = \boxed{\frac{1}{\lambda^2}}$$

An important (and distinguishing) property of exponential random variables: memorylessness!

$X \sim Exponential(\lambda)$

$$P(X \le a) = \int_0^a \lambda e^{-\lambda x}\,dx = \lambda \frac{e^{-\lambda x}}{-\lambda}\Big|_0^a = -e^{-\lambda a} + 1$$

$$= 1 - e^{-\lambda a}$$

$$P(X > a) = 1 - P(X \le a) = 1 - \left(1 - e^{-\lambda a}\right) = e^{-\lambda a}$$

$$P(X > a+b \mid X > a) = \frac{P(X > a+b \text{ AND } X > a)}{P(X>a)}$$

$a, b > 0$

$$= \frac{P(X > a+b)}{P(X>a)} = \frac{e^{-\lambda(a+b)}}{e^{-\lambda a}}$$

$$= \frac{\cancel{e^{-\lambda a}}\, e^{-\lambda b}}{\cancel{e^{-\lambda a}}} = P(X > b)$$

In other words:

$$P(X > a+b) = P(X>a)P(X>b)$$

"Waiting time" is typically modelled by exp. rand. vr.

In more concrete terms: if X = time until some event happens, and you already waited 30 min, then the probability that you will have to wait at least another 10 min is the same as the (unconditional) probability of having to wait at least 10 min starting at time 0 min.

In other words, the fact (condition) that you already waited 30 min "means nothing at all"!

$$\underbrace{P(X > 40 \mid X > 30)}_{\text{Given that it didn't happen in first 30 min, prob. that it happens between 30:00 and 40:00.}} = \underbrace{P(X > 10)}_{\text{Prob. of it happening in first 10 min (from 0:00 to 10:00)}}.$$

Exercise: Suppose the time of use it takes until a smartphone fails is modelled by an exponential random variable. The average time until failure is 1000 hours. What is the probability that the smartphone

a) fails in the first 10 hours?

b) doesn't fail in the first 1000 hours?

c) never fails?

X = time until failure [hours]. $\sim \text{Exponential}(\lambda)$

$$E(X) = 1000 \Rightarrow \lambda = \frac{1}{1000} \qquad f(x) = \frac{1}{1000} e^{-\frac{1}{1000}x}$$

a) $$P(X < 10) = \int_0^{10} \frac{1}{1000} e^{-\frac{x}{1000}} dx = \frac{1}{\cancel{1000}} \left(\frac{e^{-\frac{x}{1000}}}{-\frac{1}{\cancel{1000}}} \right) \Bigg|_0^{10}$$

$$= -e^{-\frac{1}{100}} + 1 = 1 - e^{-\frac{1}{100}} \cong 0.00995$$

$(\cong 1\%)$

b) $$P(X \geq 1000) = \int_{1000}^{+\infty} \frac{1}{1000} e^{-\frac{x}{1000}} dx$$

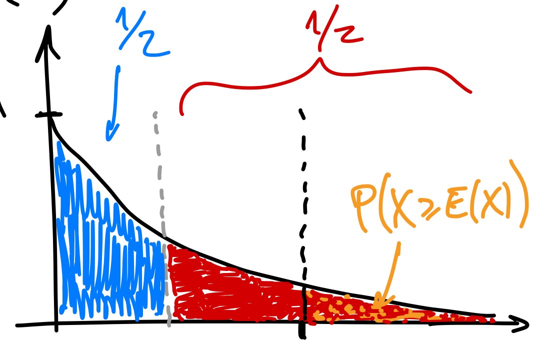
$$= 1 - \int_0^{1000} \frac{1}{1000} e^{-\frac{x}{1000}} dx$$

$$= e^{-\frac{1}{1000} \cdot 1000} = e^{-1} \cong 0.3678 \ (\cong 36.78\%)$$

c) $$P(X=+\infty) = \int_{+\infty}^{+\infty} \frac{1}{1000} e^{-\frac{x}{1000}}\,dx = 0$$

Obs: The <u>median</u> of $X \sim \text{Exponential}(\lambda)$ is

$$\frac{\ln 2}{\lambda} < \frac{1}{\lambda} = E(X)$$



Median $= \frac{\ln 2}{\lambda} < \frac{1}{\lambda} = E(X)$

As seen in the above example:

$$P(X \geq E(X)) < \frac{1}{2}$$

$$P\left(X < \frac{\ln 2}{\lambda}\right) = P\left(X > \frac{\ln 2}{\lambda}\right) = \frac{1}{2}$$

<u>Hazard Rate</u>

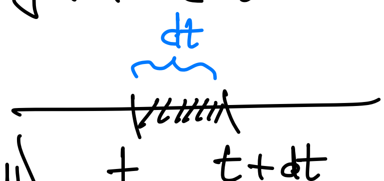
lifetime of an item or an individual

Let X be a <u>positive</u> continuous random variable, with p.d.f $f(x)$ and c.d.f $F(x)$.

$$F(x) = P(X \leq x) = \int_{-\infty}^{x} f(t)\,dt$$

FTC: $F'(x) = f(x)$.

Suppose that the item has already survived for time t. What is the prob. that it does not survive for an additional time dt.



$$P(t \leq X \leq t+dt \mid X \geq t) = \frac{P(t \leq X \leq t+dt)}{P(X \geq t)} =$$

$P(X \geq t) = 1 - P(X < t)$

$$= \frac{P(t \leq X \leq t+dt)}{1-F(t)} \overset{dt \approx 0}{\approx} \frac{f(t)}{1-F(t)}$$

Def: The <u>hazard rate</u> of X is the function

$$\lambda(t) = \frac{f(t)}{1-F(t)}$$

Note: If $X \sim \text{Exponential}(\lambda)$, then

$$f(x) = \lambda e^{-\lambda x} \qquad (x \geq 0)$$

$$F(x) = \int_0^x \lambda e^{-\lambda t}\,dt = 1 - e^{-\lambda x}$$

<u>Memorylessness:</u>

$$P(X \geq x) = 1 - F(x) = e^{-\lambda x}$$

So

$$\lambda(t) = \frac{f(t)}{1-F(t)} = \frac{\lambda e^{-\lambda x}}{e^{-\lambda x}} = \lambda.$$

Upshot: Exponential random variables have <u>constant</u> hazard rate (equal to the parameter λ).

Thm. Hazard rate uniquely determines the distribution.

"Knowing $\lambda(t)$ implies knowing $F(x)$"

Pf:

$$\int_0^t \lambda(s)\,ds = \int_0^t \frac{f(s)}{1-F(s)}\,ds = -\log(1-F(s))\Big|_0^t$$

$$= -\log(1-F(t)) + \cancel{\log(1-\underbrace{F(0)}_{=0})} \rightarrow 0$$

$$= -\log(1-F(t))$$

Therefore, solving for $F(t)$ we find

$$e^{\int_0^t \lambda(s)\,ds} = e^{\log(1-F(t))^{-1}} = \frac{1}{1-F(t)}$$

$$\Longrightarrow \boxed{F(t) = 1 - e^{-\int_0^t \lambda(s)\,ds}}$$

Cor: $f(t) = F'(t) = -e^{-\int_0^t \lambda(s)\,ds} \cdot \left(-\lambda(t)\right)$

$$= \lambda(t) e^{-\int_0^t \lambda(s)\,ds}$$

So if $\lambda(t) \equiv \lambda$, then $f(t) = \lambda e^{-\lambda t}$