## Quick recap of Calculus:

$$e^x = \sum_{n=0}^{+\infty} \frac{x^n}{n!} \quad (\text{converges for all } x \in \mathbb{R})$$

$$xe^x = x \cdot \sum_{n=0}^{+\infty} \frac{x^n}{n!} = \sum_{n=0}^{+\infty} \frac{x^{n+1}}{n!} \overset{\text{shift}}{=} \sum_{n=1}^{+\infty} \frac{x^n}{(n-1)!}$$

$$\overset{n\cdot(n-1)!=n!}{=} \sum_{n=1}^{+\infty} \frac{nx^n}{n!} \overset{0!=1}{=} \sum_{n=0}^{+\infty} \frac{nx^n}{n!}$$

Upshot: $\displaystyle\sum_{n=0}^{+\infty} \frac{x^n}{n!} = e^x, \quad \sum_{n=0}^{+\infty} \frac{nx^n}{n!} = xe^x, \quad \forall x \in \mathbb{R}.$

## Poisson Random Variables

A discrete random variable $X$ is Poisson w/ parameter $\lambda > 0$ if

$$X : \Omega \to \{0, 1, 2, 3, \dots\}$$

all values are nonnegative integers. $(\mathbb{N} \cup \{0\})$

$$P(X = i) = e^{-\lambda} \cdot \frac{\lambda^i}{i!}, \quad i = 0, 1, 2, 3, \dots$$

$p(i) = e^{-\lambda} \dfrac{\lambda^i}{i!}$ is the prob. mass function.

Write $X \sim \text{Poisson}(\lambda)$

## Expected Value:

$X \sim \text{Poisson}(\lambda)$

$$E(X) = \sum_{i=0}^{+\infty} i \cdot p(i) = \sum_{i=0}^{+\infty} i \cdot e^{-\lambda} \frac{\lambda^i}{i!} = e^{-\lambda} \underbrace{\sum_{i=0}^{+\infty} \frac{i\,\lambda^i}{i!}}_{\lambda e^{\lambda}}$$

$$= e^{-\lambda} \cdot \lambda e^{\lambda} = \boxed{\lambda.}$$

## Variance:

$$\text{Var}(X) = E(X^2) - E(X)^2 = \underbrace{E(X^2)}_{=?} - \lambda^2$$

$$E(X^2) = \sum_{i=0}^{+\infty} i^2 p(i) = \sum_{i=0}^{+\infty} i^2 e^{-\lambda} \frac{\lambda^i}{i!} = e^{-\lambda} \sum_{i=0}^{+\infty} \frac{i^2 \lambda^i}{i!}$$

$$\left( i = 0 \Rightarrow \frac{i^2 \lambda^i}{i!} = 0 \right)$$

$$= e^{-\lambda} \sum_{i=1}^{+\infty} \frac{i^2 \lambda^i}{i!} = e^{-\lambda} \sum_{i=1}^{+\infty} \frac{i\,\lambda^i}{(i-1)!} \underset{j=i-1}{=} e^{-\lambda} \sum_{j=0}^{+\infty} \frac{(j+1)\lambda^{j+1}}{j!}$$

$$= e^{-\lambda} \cdot \sum_{j=0}^{+\infty} \left( \frac{j\lambda^{j+1}}{j!} + \frac{\lambda^{j+1}}{j!} \right) = e^{-\lambda} \left( \lambda \cdot \underbrace{\sum_{j=0}^{+\infty} \frac{j\lambda^j}{j!}}_{\lambda e^{\lambda}} + \lambda \underbrace{\sum_{j=0}^{+\infty} \frac{\lambda^j}{j!}}_{e^{\lambda}} \right)$$

$$= e^{-\lambda} \left( \lambda \lambda e^{\lambda} + \lambda e^{\lambda} \right) = \lambda^2 + \lambda$$

$$\therefore \text{Var}(X) = \lambda^2 + \lambda - \lambda^2 = \boxed{\lambda.}$$

<u>Example</u>: Suppose the number of typographical errors on a single page of a book has Poisson distr. w/ parameter $\lambda = \frac{1}{2}$. What is the probability that

a) A given page contains at least 1 error?

$$X = \#\left(\begin{matrix}\text{errors in}\\ \text{a page}\end{matrix}\right) \sim \text{Poisson}\left(\tfrac{1}{2}\right) \Rightarrow P(X=i) = \frac{e^{-\frac{1}{2}}\left(\frac{1}{2}\right)^i}{i!}$$

$$P(X \geq 1) = 1 - P(X<1) = 1 - P(X=0)$$

$$= 1 - \frac{e^{-\frac{1}{2}}\left(\frac{1}{2}\right)^0}{0!} = 1 - e^{-\frac{1}{2}} = 1 - \frac{1}{\sqrt{e}} \cong 0.393$$

$(\cong 39.3\%)$

b) A given page contains no more than 4 errors?

$$P(X \leq 4) = P(X=0) + P(X=1) + P(X=2) + P(X=3) + P(X=4)$$

$$= \frac{e^{-\frac{1}{2}}\left(\frac{1}{2}\right)^0}{0!} + \frac{e^{-\frac{1}{2}}\left(\frac{1}{2}\right)^1}{1!} + \frac{e^{-\frac{1}{2}}\left(\frac{1}{2}\right)^2}{2!} + \frac{e^{-\frac{1}{2}}\left(\frac{1}{2}\right)^3}{3!} + \frac{e^{-\frac{1}{2}}\left(\frac{1}{2}\right)^4}{4!}$$

$=1$, $=\frac{1}{2}$, $=\frac{1}{4}$, $=\frac{1}{8}$, $=\frac{1}{16}$

$$= e^{-\frac{1}{2}}\left(1 + \frac{1}{2} + \frac{1}{8} + \frac{1}{48} + \frac{1}{384}\right) = e^{-\frac{1}{2}}\,\frac{211}{128}$$

$$= \frac{211}{128\sqrt{e}} \cong 0.9998 \;(\cong 99.98\%)$$

Q: What real world random variables are Poisson?

A: Those that count the number of events occurring in a fixed interval of time (or space) if these events:

- occur with a given constant mean rate
- independently of the time since the last event

Examples:

- Number of errors / defective products
- Number of large meteorites hitting Earth
- Number of incoming phone calls in a call center.

What really is $\lambda$?

If given rate is $r$, and we are interested in a time interval of length $t$, then use $\lambda = r.t$.

$$P\left(\begin{array}{c} i \text{ events in} \\ \text{time interval of} \\ \text{length } t \end{array}\right) = \frac{(rt)^i e^{-rt}}{i!}$$

Example: Suppose calls to the President's office at Lehman arrive independently and at random, with outside calls arriving at a rate of 3 in any 10 minute interval, and inside calls arriving at a rate of 4 in any 10 min. time interval. Compute the probability that 2 or more calls arrive in any 2 min. time interval.

$r = \frac{7}{10}$, $t = 2$

Sol: $X = \#$ of phone calls $\sim \text{Poisson}(\lambda)$

$$\lambda = E(X) = \underbrace{\frac{3+4}{10}}_{=r} \cdot \underbrace{2}_{=t} = \frac{7}{5} \implies P(X=i) = \frac{e^{-\frac{7}{5}} \cdot \left(\frac{7}{5}\right)^i}{i!}$$

$$P(X \geq 2) = 1 - P(X=0) - P(X=1)$$

$$= 1 - \frac{e^{-\frac{7}{5}} \cdot \left(\frac{7}{5}\right)^0}{0!} - \frac{e^{-\frac{7}{5}} \left(\frac{7}{5}\right)^1}{1!}$$

$$= 1 - e^{-\frac{7}{5}}\left(1 + \frac{7}{5}\right) = 1 - \frac{12}{5} e^{-7/5} \cong 0.4081$$

$$(= 40.81\%)$$

<u>Interesting fact</u>: Poisson random variables give a good approximation for Binomial random variables with <u>large $n$</u> and <u>moderate $n \cdot p$</u>

$Y \sim \text{Binomial}(n, p)$

$P(Y=i) = \binom{n}{i} p^i (1-p)^{n-i}$

Recall from Calculus: $\lim_{n \to \infty} \left(1 + \frac{x}{n}\right)^n = e^x$

If $n$ is very large, set $\underset{E(X)}{\lambda} = \underbrace{n \cdot p}_{E(Y)}$, then:

$X \sim \text{Poisson}(\lambda)$ satisfies

$$P(Y=i) = \binom{n}{i} p^i (1-p)^{n-i} \overset{p=\frac{\lambda}{n}}{=} \frac{n!}{i!(n-i)!} \left(\frac{\lambda}{n}\right)^i \left(1 - \frac{\lambda}{n}\right)^{n-i}$$

$$= \underbrace{\boxed{\frac{n(n-1)\dots(n-i+1)}{n^i}}}_{\approx 1} \frac{\lambda^i}{i!} \frac{\overbrace{\boxed{\left(1 - \frac{\lambda}{n}\right)^n}}^{\approx e^{-\lambda}}}{\underbrace{\boxed{\left(1 - \frac{\lambda}{n}\right)^i}}_{\approx 1}}$$

$$\approx \frac{\lambda^i}{i!} e^{-\lambda} = P(X=i). \quad \square$$