MAT330/681

Solution to the Practice Problems

1. $\binom{9}{2,2,3,2} = \frac{9!}{2!2!3!2!} = \boxed{7,560}$

2. a) $\boxed{360} = \frac{6!}{2!}$ ← repeated M's

b) $\boxed{120} = \frac{5!}{2!} \cdot 2!$ treat UE as combo letter, then permute inside it

c) $\frac{120}{360} = \boxed{\frac{1}{3}}$.

Problems 1 and 2 are similar to HW1, and exercises in Lectures 1 and 2

3. a) $\boxed{\frac{1}{2^{100}}} \cong 7.889 \cdot 10^{-31}$

b) $\boxed{\frac{1}{2}}$

c) X = number of heads $\sim \text{Binomial}(100, \frac{1}{2})$

$$P(X=50) = \binom{100}{50} \frac{1}{2^{50}} \cdot \frac{1}{2^{50}} = \boxed{\frac{100!}{50!\,50!} \frac{1}{2^{100}}} \cong 0.0796 \boxed{= 7.96\%}$$

d) X = number of heads $\sim \text{Binomial}(n, \frac{1}{2})$, $n \geq 2$

$$p_n := P(X=2) = \binom{n}{2} \frac{1}{2^2} \cdot \frac{1}{2^{n-2}} = \frac{n!}{2!(n-2)!} \cdot \frac{1}{2^n} = \frac{n(n-1)}{2^{n+1}}$$

$p_2 = \frac{1}{4}$, $\boxed{p_3 = \frac{3}{8}, \quad p_4 = \frac{3}{8}}$, $p_5 = \frac{5}{16}$, $p_6 = \frac{15}{64}$, $p_7 = \frac{21}{128} \ldots$

Note p_n is maximal for $n=3$ and $n=4$, since

$$p_n < p_{n+1} \iff \frac{n(n-1)}{2^{n+1}} < \frac{(n+1)n}{2^{n+2}} \iff 2(n-1) < n+1 \iff n < 3$$

$$p_n > p_{n+1} \iff \frac{n(n-1)}{2^{n+1}} > \frac{(n+1)n}{2^{n+2}} \iff 2(n-1) > n+1 \iff n > 3$$

So choose either $n=3$ or $n=4$, since

$$p_3 = p_4 = \frac{3}{8} > p_n \quad \text{for all} \quad n \neq 3, 4.$$

← Similar to HW6 #1 and exercises in Lecture 11 (e.g. Video 3)

4. $X \sim \text{Binomial}(p, 10)$.

$$P(X \geq 7) = P(X=7) + P(X=8) + P(X=9) + P(X=10)$$

$$= \binom{10}{7} p^7 (1-p)^3 + \binom{10}{8} p^8 (1-p)^2 + \binom{10}{9} p^9 (1-p) + \binom{10}{10} p^{10}$$

$$= 120\, p^7 (1-p)^3 + 45\, p^8 (1-p)^2 + 10\, p^9 (1-p) + p^{10}.$$

5.

X	1	4	5
$P(X=x_i)$	$1/4$	$1/2$	$1/4$

← Similar to exercises in Lecture 9

a) $P(X>3) = P(X=4) + P(X=5) = 1/2 + 1/4 = 3/4 = 75\%$

b) $$P(X>3 \mid X>2) = \frac{P(X>3 \text{ and } X>2)}{P(X>2)} = \frac{P(X>3)}{P(X>2)} = 1 = 100\%$$

b/c $X>2 \iff X>3$, as X only assumes values $\{1, 4, 5\}$.

c) $$P(X>2 \mid X>3) = \frac{P(X>2 \text{ and } X>3)}{P(X>3)} = \frac{P(X>3)}{P(X>3)} = 1 = 100\%$$

d) $$E(X) = \frac{1}{4} \cdot 1 + \frac{1}{2} \cdot 4 + \frac{1}{4} \cdot 5 = \frac{7}{2} = 3.5$$

e) $$E(X^2) = \frac{1}{4} \cdot 1^2 + \frac{1}{2} \cdot 4^2 + \frac{1}{4} \cdot 5^2 = \frac{29}{2} = 14.5$$

f) $\mathrm{Var}(X) = E(X^2) - E(X)^2 = \frac{29}{2} - \left(\frac{7}{2}\right)^2 = \boxed{\frac{9}{4} = 2.25}$

g) $E(\sqrt{X}) = \frac{1}{4}\cdot\sqrt{1} + \frac{1}{2}\sqrt{4} + \frac{1}{4}\sqrt{5} = \frac{1}{4} + 1 + \frac{\sqrt{5}}{4} = \boxed{\frac{5+\sqrt{5}}{4}}$

Similar to exercises in Lecture 4 and HW2 #1

6. By inclusion-exclusion:

$$P(A\cup B\cup C) = P(A) + P(B) + P(C) - P(AB) - P(AC) - P(BC) + P(ABC)$$

$$= P(A) + P(B) + P(C) - P(A)P(B) - P(A)P(C) - P(B)P(C) + P(A)P(B)P(C).$$

$$\left\{\begin{array}{l} P(A) = 0.4 \\ P(B) = 0.3 \\ P(C) = 0.25 \end{array}\right. \Rightarrow \boxed{P(A\cup B\cup C) = 0.685 = 68.5\%}$$

Same problem as HW4 #3.

7. E = 4 out of 10 plants grew

H = farmer used the non-expired fertilizer

$P(H) = P(H^c) = \frac{1}{2}$, $\quad P(E|H) = \binom{10}{4}\left(\frac{2}{3}\right)^4\left(\frac{1}{3}\right)^6 \leftarrow$ 2/3 grow w/ non-expired fertilizer

$P(E|H^c) = \binom{10}{4}\left(\frac{1}{3}\right)^4\left(\frac{2}{3}\right)^6 \leftarrow$ 1/3 grow w/ expired fertilizer

$$P(H|E) = \frac{P(E|H)\overbrace{P(H)}^{=\frac{1}{2}}}{P(E|H)\underbrace{P(H)}_{=\frac{1}{2}} + P(E|H^c)\underbrace{P(H^c)}_{=\frac{1}{2}}} = \frac{\cancel{\binom{10}{4}}\left(\frac{2}{3}\right)^4\left(\frac{1}{3}\right)^6}{\cancel{\binom{10}{4}}\left(\frac{2}{3}\right)^4\left(\frac{1}{3}\right)^6 + \cancel{\binom{10}{4}}\left(\frac{1}{3}\right)^4\left(\frac{2}{3}\right)^6}$$

$$= \frac{1}{1 + \frac{\left(\frac{1}{3}\right)^4\left(\frac{2}{3}\right)^6}{\left(\frac{2}{3}\right)^4\left(\frac{1}{3}\right)^6}} = \frac{1}{1 + \frac{2^6}{2^4}} = \frac{1}{1+4} = \boxed{\frac{1}{5} = 20\%}$$

Similar to HW5 #2.

8. X = Fine received each time commuter speeds

$P(X=0) = 0.1$ ← not caught

$P(X=200) = 0.9$ ← caught

$E(X) = 0.9 \cdot 200 + 0.1 \cdot 0 = 180. \implies \boxed{E(5X) = 5 \cdot 180 = 900}$

Same problem as HW7 #1 and similar to exercises in Lecture 12, Video 3.

9. X = # packages arriving at distribution center

65 packages/10 min = 6.5 packages/min

$X \sim \text{Poisson}(\lambda)$, $\lambda = E(X) = \underset{\text{rate}}{r} \cdot \underset{\text{time}}{t} = \frac{65}{10} \cdot 2 = 13$ ← time interval of 2 min

$$\boxed{P(X=4) = \frac{e^{-\lambda}\lambda^4}{4!} = \frac{e^{-13}\, 13^4}{24}} \cong 0.00269 = 0.27\%$$

Similar to exercises in Lecture 15

10. X = life time of solar panel

$X \sim \text{Exponential}(\lambda)$ $E(X) = \frac{1}{\lambda} = 200 \implies \lambda = \frac{1}{200}$

$P(X \geq 100) = e^{-\frac{100}{200}} = e^{-1/2} = \frac{1}{\sqrt{e}}$ ← call this p.

$$P\left(\begin{array}{l}\text{At least 1 of 3 panels} \\ \text{lasts} \geq 100 \text{ months}\end{array}\right) = \binom{3}{1} p (1-p)^2 + \binom{3}{2} p^2 (1-p) + \binom{3}{3} p^3$$

$$= 3p(1-p)^2 + 3p^2(1-p) + p^3$$

$$= 3p(1-p) + p^3$$

$$= 1 - (1-p)^3.$$

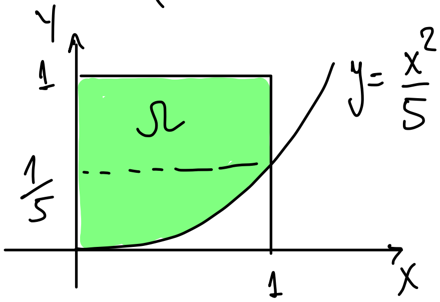
$p = \frac{1}{\sqrt{e}}$

$$\cong \boxed{1 - \left(1 - \frac{1}{\sqrt{e}}\right)^3} \cong 0.939 = 93.9\%$$

Similar to HW7 #2, 3

11. $X, Y \sim \mathrm{Unif}([0,1])$

$$P(X^2 < 5Y) = \text{Area}(\Omega) = 1 - \int_0^1 \frac{x^2}{5}\,dx = 1 - \frac{1}{5}\frac{x^3}{3}\Big|_0^1 = 1 - \frac{1}{15} = \boxed{\frac{14}{15}}$$



Same as Lecture 15, Video 5, similar also to HW8 #2.

12. $\lambda(t) = 3t^2 \Rightarrow F(t) = 1 - e^{-\int_0^t 3s^2\,ds} = 1 - e^{-t^3}$

$$\Rightarrow f(t) = F'(t) = 3t^2 e^{-t^3} \Rightarrow \boxed{f(x) = 3x^2 e^{-x^3}}$$

Same as HW11 #2 and exercises in Lecture 21

13. X = # letters

Y = # numbers

Z = # special characters

a) $$P(X \geq 8, Y \geq 3, Z \geq 3) \overset{\text{indep}}{=} P(X \geq 8) \cdot P(Y \geq 3) \cdot P(Z \geq 3)$$

$$\underset{\text{Markov}}{\leq} \frac{E(X)}{8} \cdot \frac{E(Y)}{3} \cdot \frac{E(Z)}{3}$$

$$= \frac{\cancel{6}}{\cancel{8}} \cdot \frac{\cancel{2}}{\cancel{3}} \cdot \frac{\cancel{2}}{3} = \boxed{\frac{1}{3}}$$

b) $$P(3 \leq X \leq 9) = P(|X - 6| \leq 3)$$

$$= 1 - P(|X - 6| \geq 3)$$

$$\underset{\text{Chebyshev}}{\geq} 1 - \frac{1^2}{3^2} = \boxed{\frac{8}{9}}$$

Similar to Video 2 in Lecture 22 and HW12 #2

14. X_i = return from i^{th} trade ← iid w/ $\mu = 2$, $\sigma = 0.5$

$$P\left(190 \le X_1 + \dots + X_{100}\right) = P\left(\frac{190}{100} \le \overline{X}_{100}\right)$$

$$= P\left(\frac{19}{10} - 2 \le \overline{X}_{100} - \mu\right)$$

$$= P\left(\frac{-\frac{1}{10}}{\sigma/\sqrt{100}} \le \frac{\overline{X}_{100} - \mu}{\sigma/\sqrt{100}}\right)$$

$n = 100$,
$\mu = 2$, $\sigma = 0.5$

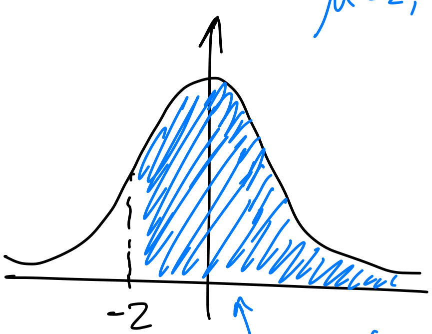
$$= P\left(-2 \le \frac{\overline{X}_{100} - 2}{0.5/10}\right)$$

CLT

$$\cong P\left(-2 \le Z\right)$$

$$= \Phi(2) = \boxed{0.9772}$$

Table



$= \Phi(2)$

Similar to video 3 in Lecture 22

15. X_i = result of i^{th} measurement (i^{th} patient) ← iid w/ $\mu = 5$ and $\sigma = 3$.

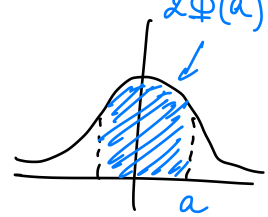
Want n large enough so that

$$P\left(|\overline{X}_n - 5| \le 0.5\right) \ge \frac{92}{100}.$$

But

$$P\left(|\overline{X}_n - 5| \le 0.5\right) = P\left(-0.5 \le \overline{X}_n - 5 \le 0.5\right)$$

$2\Phi(a) - 1$



CLT

$$= P\left(\frac{-0.5}{3/\sqrt{n}} \le \frac{\overline{X}_n - 5}{3/\sqrt{n}} \le \frac{0.5}{3/\sqrt{n}}\right) \cong P\left(-\frac{\sqrt{n}}{6} \le Z \le \frac{\sqrt{n}}{6}\right)$$

$$= 2 \cdot \Phi\left(\frac{\sqrt{n}}{6}\right) - 1$$

So, want:

$$\frac{92}{100} \leq 2\Phi\left(\frac{\sqrt{n}}{6}\right) - 1 \iff \Phi\left(\frac{\sqrt{n}}{6}\right) \geq \frac{192}{200} = \frac{24}{25} = 0.96$$

By Table, $\Phi(1.75) = 0.9599$, so $\frac{\sqrt{n}}{6} \geq 1.75$, i.e.

$$n \geq 36 \cdot 1.75^2 = 110.25, \quad \text{so} \quad n \geq 111.$$

Need at least 111 patients.