Name: ANSWERS **Lehman ID:** ______________

MAT 330/681
Final Exam
May 18, 2022

Instructions:
Turn off and put away your cell phone.
Please write your Name and Lehman ID # on the top of this page.
Please sign and date the pledge below to comply with the Code of Academic Integrity.
You are allowed to consult a 1-page reference sheet (written on both sides) and use a simple calculator which is not a smartphone. No other consultation materials or electronic devices are allowed during the exam.
If any question is unclear, raise your hand to ask for clarifications.
The regular amount of time you have to complete the exam is 120 minutes.
You must show all of your work! *No credit will be given for unsupported answers.*
Please try to be as organized, objective, and logical as possible in your answers.

#	**Points**	**Score**
1	10	
2	10	
3	15	
4	15	
5	10	
6	15	
7	10	
8	15	
Total	100	

My signature below certifies that I complied with the CUNY Academic Integrity Policy and the Lehman College Code of Academic Integrity in completing this examination.

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Signature Date

Problem 1 (10 pts): You are placing flowes in line along a wall on the side of your house, and you will plant 4 tulips, 5 roses, 3 daffodiles and 2 sunflowers.

a) **(5 pts)** How many different arrangements can you plant with those flowers?

$$\binom{14}{4,5,3,2} = \frac{14!}{4!5!\,3!\,2!} = 2{,}522{,}520$$

b) **(5 pts)** In how many arrangements are all tulips planted next to each other?

Treat all 4 tulips as 1 group:

$$\binom{11}{1,5,3,2} = \frac{11!}{1!5!\,3!\,2!} = 27{,}720$$

Problem 2 (10 pts): A building has 15 floors: 1 (ground), 2, 3, ..., 15. Three people walk in to the elevator on the ground floor and each presses the button of the floor they are going to. What is the probability that 3 consecutive buttons (for example: 2, 3, 4; or 10, 11, 12) are pressed?

Hint: Keep in mind that nobody entering the elevator on the ground floor is going to the ground floor itself.

Consecutive floors:
$$\left.\begin{matrix} 2,3,4 \\ 3,4,5 \\ \vdots \\ 13,14,15 \end{matrix}\right\} \text{12 possibilities (ordered)}$$

in any order

$$P = \frac{12 \cdot 3!}{14^3} = \frac{9}{343} \cong 2.62\%$$

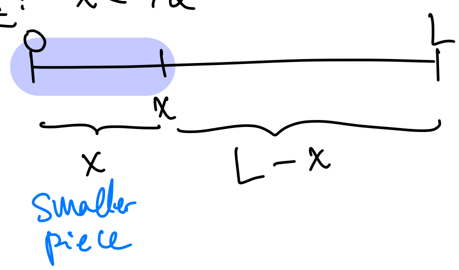
total possibilities

$\underbrace{\quad}_{\text{any floor}}\underbrace{\quad}_{\text{any floor}}\underbrace{\quad}_{\text{any floor}}$

Problem 3 (15 pts): You break a stick at a uniformly chosen random location. What is the probability that the shorter piece is less than 1/5th of the original?

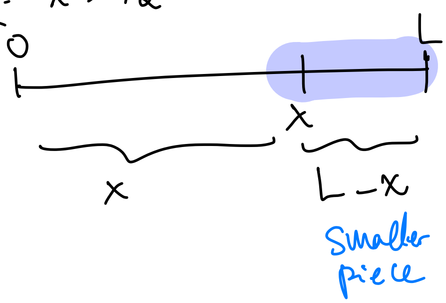
$X \sim \text{Uniform}([0,L])$ break point, length L

Case 1: $x < L/2$



$$P\left(X < \frac{L}{5}\right) = \frac{1}{5}$$

Case 2: $x > L/2$



$$P\left(1 - x < \frac{L}{5}\right) = P\left(X > \frac{4L}{5}\right) = \frac{1}{5}$$

$$P\left(\begin{matrix}\text{smaller piece} \\ \text{is } < \frac{1}{5}\end{matrix}\right) = \frac{1}{5} + \frac{1}{5} = \frac{2}{5} = 40\%$$

Problem 4 (15 pts): The probability that a visit to a primary care physician's (PCP) office results in neither lab work nor referral to a specialist is 35%. Of those coming to a PCP's office, 30% are referred to specialists and 40% require lab work. What is the probability that a visit to a PCP's office results in both lab work and referral to a specialist?

L = need lab work

R = need referral to specialist

$P(L^c \cap R^c) = 0.35, \quad P(R) = 0.3, \quad P(L) = 0.4$

$$
\begin{aligned}
P(L \cap R) &= 1 - P((L \cap R)^c) \\
&= 1 - P(L^c \cup R^c) \\
&= 1 - \big(P(L^c) + P(R^c) - P(L^c \cap R^c)\big) \\
&= 1 - (0.6 + 0.7 - 0.35) \\
&= 0.05 = 5\%
\end{aligned}
$$

Problem 5 (10 pts): Suppose $X \sim$ Exponential$(1/3)$. Compute the following:

a) **(5 pts)** $P(1 \leq X < 8) =$ $F(8) - F(1)$

$$= \left(1 - e^{-8/3}\right) - \left(1 - e^{-1/3}\right)$$

$$= e^{-1/3} - e^{-8/3} \simeq 64.7\%$$

b) **(5 pts)** $P(X > 4 | X > 1) =$ $\dfrac{P(X>4 \ \& \ X>1)}{P(X>1)} = \dfrac{P(X>4)}{P(X>1)}$

$$= \frac{e^{-4/3}}{e^{-1/3}} = e^{-1} = \frac{1}{e} \cong 36.79\%$$

Alternatively,

$$P(X>4|X>1) \underset{\substack{\uparrow \\ \text{memorylessness}}}{=} P(X>3) = e^{-3/3} = \frac{1}{e}.$$

Problem 6 (15 pts): Someone proposes you the following game: three fair die with faces labeled 1 through 6 are tossed, and you win a prize of \$5.00 if at least two of them land on faces 5 or 6, but you lose \$4.00 otherwise.

a) **(10 pts)** What is the expected profit (or loss) of playing this game?

$$P\left(\begin{matrix}\text{Dice lands}\\ \text{on 5 or 6}\end{matrix}\right) = \frac{1}{6} + \frac{1}{6} = \frac{1}{3}$$

$$P\left(\begin{matrix}\text{At least 2 of 3}\\ \text{dice land on 5 or 6}\end{matrix}\right) = \binom{3}{2}\left(\frac{1}{3}\right)^2\left(\frac{2}{3}\right)^1 + \binom{3}{3}\left(\frac{1}{3}\right)^3 = 3\cdot\frac{1}{9}\cdot\frac{2}{3} + \frac{1}{27}$$

$$= \frac{2}{9} + \frac{1}{27} = \frac{7}{27}$$

X = return from game

$$E(X) = \frac{7}{27}\cdot(+5.00) + \frac{20}{27}(-4.00) = \frac{35-80}{27} = -\frac{45}{27} = -\frac{5}{3}$$

$$\underbrace{\simeq -1.67}_{\text{expected loss of \$1.67}}$$

b) **(5 pts)** What is the value of the prize that would make this game fair?

Hint: A game is fair if you do not expect to either *win* nor *lose* any money.

$$0 = E(X) = \frac{7}{27}v + \frac{20}{27}(-4.00)$$

$$= \frac{7v-80}{27} \iff 7v = 80 \iff v = \frac{80}{7} \simeq 11.43$$

Fair prize would be \$11.43

Problem 7 (10 pts): Suppose $X_i \sim \text{Uniform}([0,1])$, $i = 1, \ldots, 200$, are independent identically distributed random variables. Use the Central Limit Theorem to approximate the following probabilities in terms of $\Phi(a) = P(Z \le a)$, where $Z \sim \text{Normal}(0,1)$.

Hint: Your answer can use the function $\Phi(a)$ more than once, for different values of a.

a) **(5 pts)** $P(X_1 + \cdots + X_{200} > 90) = P\left(\overline{X_{200}} > \frac{90}{200}\right)$

$$= P\left(\overline{X_{200}} - 1/2 > \frac{9}{20} - \frac{1}{2}\right) \qquad \frac{1}{2\sqrt{3}\,10\sqrt{2}}$$

$X_i \sim \text{Uniform}([0,1])$

$\Downarrow$

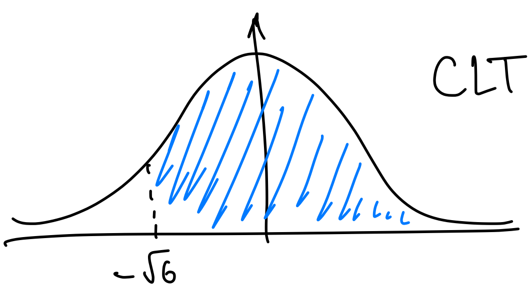
$E(X_i) = \mu = 1/2$

$Var(X_i) = \sigma^2 = 1/12$

$\sigma = 1/2\sqrt{3}$

$$= P\left(\frac{\overline{X_{200}} - 1/2}{1/2\sqrt{3}/\sqrt{200}} > \frac{-1/20}{1/2\sqrt{3}/\sqrt{200}}\right) \qquad \frac{1/2\sqrt{3}}{\sqrt{200}} = \frac{1}{20\sqrt{6}}$$

$$= P\left(\frac{\overline{X_{200}} - \mu}{\sigma/\sqrt{n}} > -\sqrt{6}\right)$$



CLT

$$\approx P\left(Z > -\sqrt{6}\right) = 1 - \Phi(\sqrt{6})$$

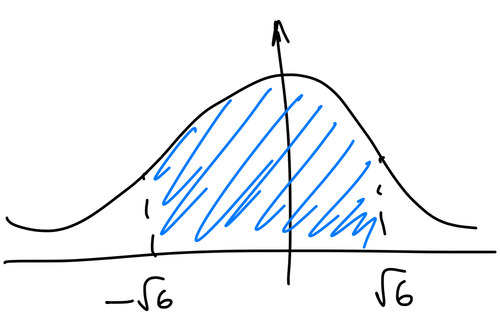
b) **(5 pts)** $P(90 < X_1 + \cdots + X_{200} < 110) = P\left(\frac{90}{200} < \overline{X_{200}} < \frac{110}{200}\right)$

$$= P\left(\frac{9}{20} - \frac{1}{2} < \overline{X_{200}} - \mu < \frac{11}{20} - \frac{1}{2}\right)$$

$$= P\left(\frac{-1/20}{1/2\sqrt{3}/\sqrt{200}} < \frac{\overline{X_{200}} - \mu}{\sigma/\sqrt{n}} < \frac{1/20}{1/2\sqrt{3}/\sqrt{200}}\right)$$

CLT

$$\approx P\left(-\sqrt{6} < Z < \sqrt{6}\right)$$



$$= \Phi(\sqrt{6}) - (1 - \Phi(\sqrt{6}))$$

$$= 2\Phi(\sqrt{6}) - 1.$$

Problem 8 (~~10~~ 15 pts): Two envelopes with a total \$300 are shown to you. One envelope contains \$100, and the other contains \$200, but you do not know which is which.

a) **(5 pts)** You choose one between the two envelopes, with equal probabilities. What is the expected value of your winnings?

X = return

$$E(X) = \frac{1}{2} \cdot 100 + \frac{1}{2} 200 = 150.$$

b) **(5 pts)** After choosing one of the two envelopes, but before opening it, you are given the opportunity to change your choice to the other envelope. Is the expected value of your winnings increased by switching? If so, by how much?

No, it is not increased by switching, because the problem is totally symmetric:

No switch: $E(X_{stay}) = \frac{1}{2} 100 + \frac{1}{2} 200 = 150$

$\Vert \leftarrow$ same!

Switch: $E(X_{switch}) = \frac{1}{2} 200 + \frac{1}{2} 100 = 150$

c) (5pts) Note this is different from Monty Hall! No information is learnt from "opening the 3rd door".