Conditional Probability

Def: $P(E|F) = \dfrac{P(E\cap F)}{P(F)} = \dfrac{P(EF)}{P(F)}$

"given that"

If S is finite and all outcomes are equally likely:

$P(E) = \dfrac{|E|}{|S|}$ $\quad P(F) = \dfrac{|F|}{|S|}$ $\quad P(EF) = \dfrac{|E\cap F|}{|S|}$

$$P(E|F) = \frac{P(EF)}{P(F)} = \frac{|E\cap F|/|S|}{|F|/|S|} = \frac{|E\cap F|}{|F|}.$$

Ex. 2 (fair) coins are flipped. What is the prob. of both landing on heads, given that:

a) first coin lands on heads?

b) at least one coin lands on heads?

$S = \{(t,t), (t,h), (h,t), (h,h)\}$ $\quad |S| = 4$

a) $P(\underbrace{(h,h)}_{\text{Both } h} \mid \underbrace{(h,t)\cup(h,h)}_{\text{First is } h}) = \dfrac{P(\boxed{(h,h)\cap[(h,t)\cup(h,h)]})}{P((h,t)\cup(h,h))}$ $\quad \boxed{\;} = (h,h)$

$$= \frac{P((h,h))}{P((h,t)\cup(h,h))} = \frac{1}{2}$$

b) $P\Big((h,h) \mid (t,h) \cup (h,t) \cup (h,h)\Big) = \dfrac{P\big((h,h)\big)}{P\big((t,h) \cup (h,t) \cup (h,h)\big)} = \dfrac{1}{3}.$

Ex: If 18% of Lehman students play tennis and baseball, and 32% ——〃—— tennis, what is the prob. that a student plays baseball, given that this student plays tennis?

T = Student plays tennis

B = Student plays baseball

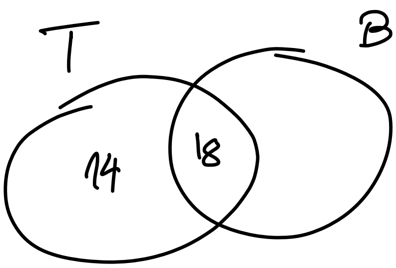
$$P(T) = \frac{32}{100}, \quad P(TB) = \frac{18}{100}. \qquad P(B|T) = ?$$

$$P(B|T) = \frac{P(TB)}{P(T)} = \frac{\frac{18}{\cancel{100}}}{\frac{32}{\cancel{100}}} = \frac{9}{16} = 56.25\%$$

$$P(T|B) = \frac{P(TB)}{P(B)}$$

we don't know this.

$$P(B^c|T) = \frac{P(B^cT)}{P(T)} = \frac{14/100}{32/100} = \frac{14}{32} = \frac{7}{16} = 1 - \frac{9}{16}$$



$TB^c = B^cT$

$P(B^cT) = 14/100$

$P : \mathcal{P}(S) \to [0,1]$

Thm: $P(\cdot\,|E)$ is a prob.

Multiplication rule

Suppose we have events $E_1, E_2, \ldots, E_n$.

$$P(E_1 E_2 \ldots E_n) = P(E_1) \cdot P(E_2 | E_1) \cdot P(E_3 | E_1 E_2) \cdots P(E_n | E_1 \ldots E_{n-1})$$

Pf:

$$P(E_1) \cdot \underbrace{P(E_2|E_1)}_{\frac{P(E_1E_2)}{P(E_1)}} \cdot \underbrace{P(E_3|E_1E_2)}_{\frac{P(E_1E_2E_3)}{P(E_1E_2)}} \cdots \underbrace{P(E_n|E_1\ldots E_{n-1})}_{\frac{P(E_1\ldots E_n)}{P(E_1\ldots E_{n-1})}} =$$

$$= \cancel{P(E_1)} \, \frac{\cancel{P(E_1E_2)}}{\cancel{P(E_1)}} \cdot \frac{\cancel{P(E_1E_2E_3)}}{\cancel{P(E_1E_2)}} \cdots \frac{P(E_1 \ldots E_n)}{\cancel{P(E_1 \ldots E_{n-1})}} = P(E_1 \ldots E_n). \qquad \square$$

Ex: A standard deck with 52 cards is randomly divided into 4 piles (with 13 cards each). What is the prob. that each pile contains <u>exactly</u> 1 ace?

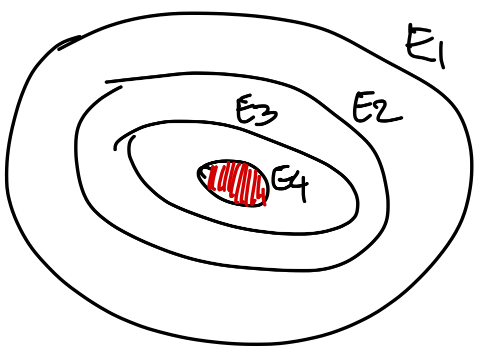
$E_1 = \{$A of spades going on some pile$\}$

$E_2 = \{$A of hearts and A of spades go in different piles$\}$

$E_3 = \{$A of clubs, A hearts, and A spades go in different piles$\}$

$E_4 = \{$All A's go in different piles$\}$.

$$E_4 \subset E_3 \subset E_2 \subset E_1$$



$$P(E_1) = 1$$

$$P(E_2|E_1) = 1 - \overbrace{P(E_2^c|E_1)}^{=\frac{12}{51}} = 1 - \frac{12}{51}.$$

$$P(E_3|E_1E_2) = 1 - P(E_3^c|E_1E_2) = 1 - \frac{24}{50}$$

$$P(E_4|E_1E_2E_3) = 1 - \frac{36}{49}$$

$$P(E_4) = P(E_1E_2E_3E_4)$$

mult. rule

$$\stackrel{\text{mult. rule}}{=} P(E_1) \cdot P(E_2|E_1) \cdot P(E_3|E_1E_2) \cdot P(E_4|E_1E_2E_3)$$

$$= 1 \cdot \left(1 - \frac{12}{51}\right) \cdot \left(1 - \frac{24}{50}\right) \cdot \left(1 - \frac{36}{49}\right) = \frac{2197}{20825} \cong 10.55\%$$

<u>Alternative Sol.</u>:

$$P = \frac{\binom{13}{1} \cdot \binom{13}{1} \cdot \binom{13}{1} \cdot \binom{13}{1}}{\binom{52}{4}} = \frac{13^4}{\binom{52}{4}} = \frac{2197}{20825}.$$

Ex: You lost your keys and are 80% sure they are in one of 2 pockets in your coat, being 40% sure they are on the Left pocket, and 40% sure they are on the Right pocket. If you look in the Left pocket and they are <u>not</u> there, what is the prob. you find them in the Right?

L = Keys are in left pocket

R = Keys are in right pocket.

$$P(L) = P(R) = \frac{40}{100} = \frac{2}{5}. \qquad P(R|L^c) = ?$$

$$P(R|L^c) = \frac{P(RL^c)}{P(L^c)} = \frac{P(R)}{1-P(L)} = \frac{2/5}{1-2/5} = \frac{2/5}{3/5} = \boxed{\frac{2}{3}}$$