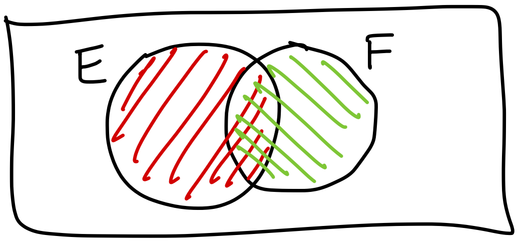
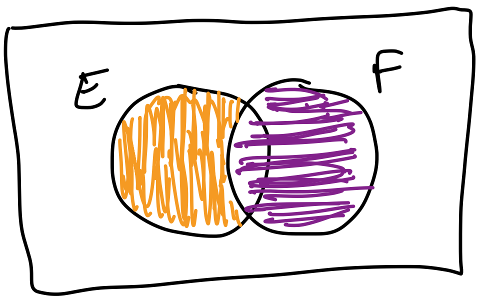
Inclusion-Exclusion principle:



$$\boxed{P(E \overset{\text{or}}{\cup} F) = P(E) + P(F) - P(E \overset{\text{and}}{\cap} F)}$$

Pf: $E \cup F = \underline{(E \cap F^c)} \cup \underline{F}$ ← disjoint!



$$P(E \cup F) = P((E \cap F^c) \cup F)$$
$$\overset{(iii)}{=} P(E \cap F^c) + P(F) \quad (1)$$

$$(E \cap F^c) \cup (E \cap F) = E \overset{(iii)}{\Longrightarrow} \underbrace{P((E \cap F^c) \cup (E \cap F))}_{P(E \cap F^c) + P(E \cap F)} = P(E)$$

$$\Longrightarrow P(E \cap F^c) = P(E) - P(E \cap F) \quad (2)$$

By (1) & (2), we conclude $P(E \cup F) = P(E) + P(F) - P(E \cap F)$. □

Amazon's artificial intelligence predicts there is a:

- 50% chance you will like book A
- 40% chance you will like book B
- 30% chance you will like both books A and B.

What is the implied chance you do not like either?

A = event that you like book A — $P(A) = \frac{1}{2}$

B = ——— " ——— B — $P(B) = \frac{2}{5}$

$A \cap B$ = ——— " ——— A and B — $P(A \cap B) = \frac{3}{10}$

$$P((A\cup B)^c) = P(A^c \cap B^c) = ?$$

$$P(A\cup B) = P(A) + P(B) - P(A\cap B)$$

$$= \frac{1}{2} + \frac{2}{5} - \frac{3}{10} = \frac{5+4-3}{10} = \frac{6}{10} = \frac{3}{5}$$

$$P((A\cup B)^c) = 1 - P(A\cup B) = 1 - \frac{3}{5} = \frac{2}{5} \quad (40\%)$$

Note: $P(E^c) = 1 - P(E)$.

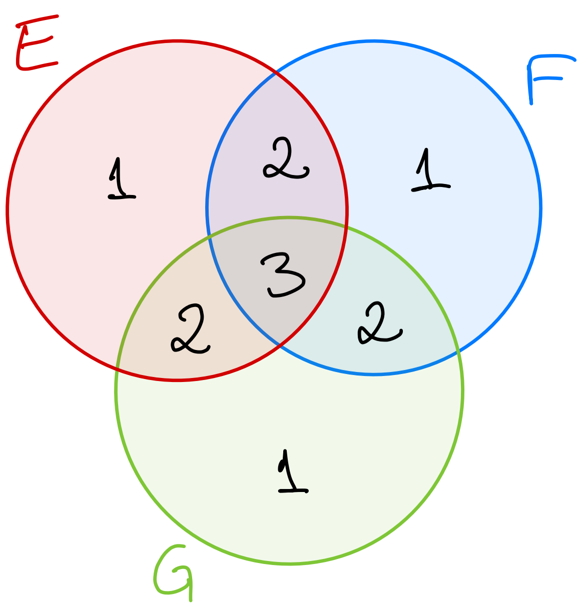
Pf: $P(E^c \cup E) = P(S) = 1$

$\quad\; \parallel$

$P(E^c) + P(E)$

$\Longrightarrow P(E^c) = 1 - P(E)$.

Inclusion-Exclusion principle with 3 or more sets:



$$P(E\cup F\cup G) = P(E) + P(F) + P(G) - P(E\cap F) - P(E\cap G) - P(F\cap G) + P(E\cap F\cap G)$$

With more sets: $E_1, E_2, E_3, \dots, E_n$:

$$P\left(\bigcup_{i=1}^{n} E_i\right) = \sum_{i_1} (-1)^2 P(E_i) + \sum_{i_1 < i_2} (-1)^3 P(E_{i_1} \cap E_{i_2})$$

$$+ \sum_{i_1<i_2<i_3} (-1)^4 P(E_{i_1} \cap E_{i_2} \cap E_{i_3}) + \dots$$

$$\dots + \sum_{i_1<i_2<\dots<i_k} (-1)^{k+1} P(E_{i_1} \cap \dots \cap E_{i_k}) \dots + (-1)^{n+1} P(E_1 \cap \dots \cap E_n)$$

$$\boxed{P\left(\bigcup_{i=1}^{n} E_i\right) = \sum_{k=1}^{n} (-1)^{k+1} \sum_{i_1<\dots<i_k} P(E_{i_1} \cap \dots \cap E_{i_k})}$$

<u>Matching problem:</u>

When classes were in person, n students were always on their phones, so the professor collected the phones and placed them in a bucket. Suppose all phones were identical models, and couldn't be distinguished from the outside. At the end of class, when these n students pick a phone at random from the bucket, what is the probability that <u>no one</u> picks their own phone?

- E_i = i^{th} student picking their own phone.
- $E_1 \cup E_2 \cup \dots \cup E_n \leftarrow$ at least someone picks up their own phone
- $(E_1 \cup E_2 \cup \dots \cup E_n)^c = E_1^c \cap E_2^c \cap \dots \cap E_n^c \leftarrow$ desired probability.

- Inclusion-exclusion principle:

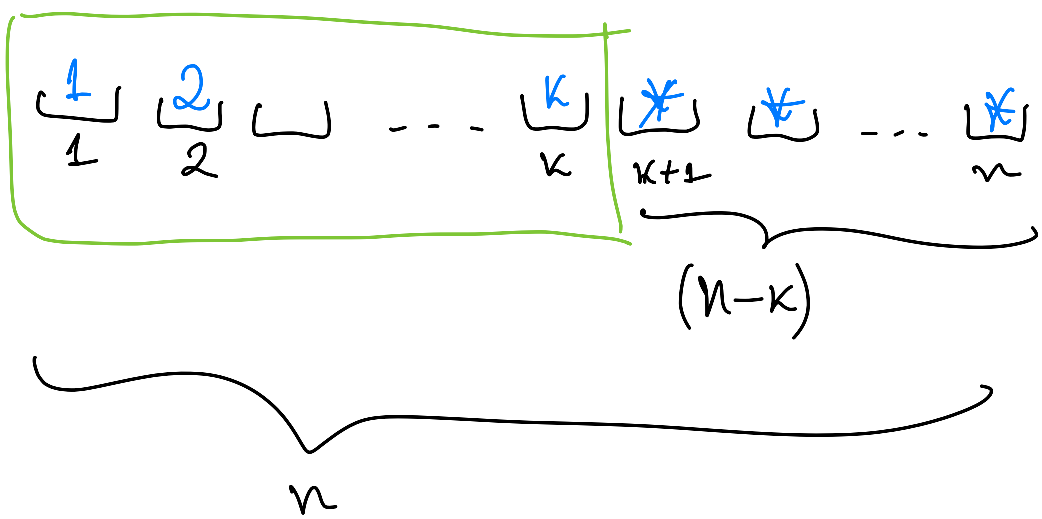
$$P(E_1 \cup E_2 \cup \dots \cup E_n) = \sum_{k=1}^{n} (-1)^{k+1} \sum_{i_1 < \dots < i_k} \underbrace{P(E_{i_1} \cap \dots \cap E_{i_k})}_{?}$$

$$P(E_{i_1} \cap \dots \cap E_{i_k}) = \frac{(n-k)!}{n!}$$

$(n-k)!$ ← permutations where $i_1 \dots i_k$ select their own phone.

$n!$ ← total # of permutations

(WLOG: $i_1 = 1$, $i_2 = 2$, ..., $i_k = k$)



How many permutations of 1 2 3 ... n fix the first k slots?

$(n-k)!$

$$P(E_1 \cup E_2 \cup \dots \cup E_n) = \sum_{k=1}^{n} (-1)^{k+1} \sum_{i_1 < \dots < i_k} \underbrace{P(E_{i_1} \cap \dots \cap E_{i_k})}_{\frac{(n-k)!}{k!}}$$

There are $\binom{n}{k}$ collections of indices $i_1 < \dots < i_k$

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}$$

$$= \sum_{k=1}^{n} (-1)^{k+1} \binom{n}{k} \frac{(n-k)!}{n!} = \sum_{k=1}^{n} \frac{(-1)^{k+1}}{k!}$$

$$P\left(\left(\bigcup_{i=1}^{n} E_i\right)^c\right) = 1 - P\left(\bigcup_{i=1}^{n} E_i\right) = 1 - \left(\underset{\uparrow \atop k=1}{\frac{1}{1}} - \frac{1}{2!} + \frac{1}{3!} - \dots + \underset{\uparrow \atop k=n}{\frac{(-1)^{n+1}}{n!}}\right)$$

$$= \frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!} - \dots + \frac{(-1)^n}{n!}$$

$$= \sum_{j=2}^{n} \frac{(-1)^j}{j!} = \sum_{j=0}^{n} \frac{(-1)^j}{j!}$$

$$j=0: \quad \frac{(-1)^0}{0!} = 1$$

$$j=1: \quad \frac{(-1)^1}{1!} = -1$$

Upshot:

A: $p(n) = \displaystyle\sum_{j=0}^{n} \frac{(-1)^j}{j!}$

$n=5$

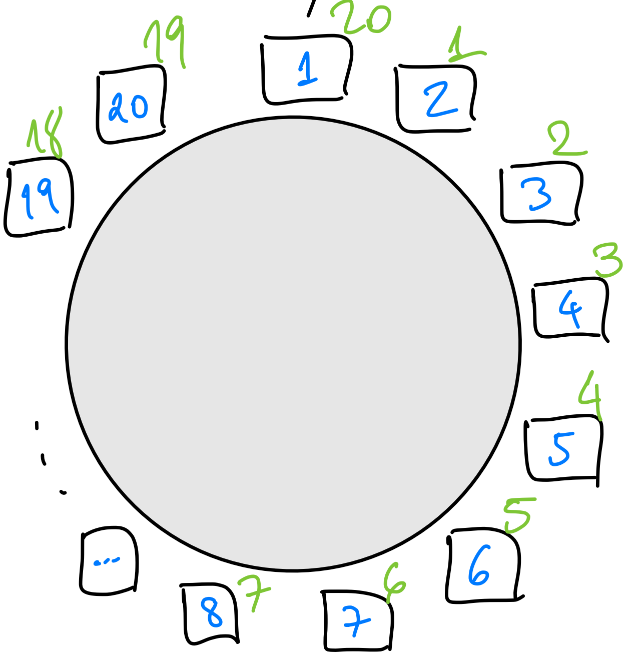
$$p(5) = \frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!} - \frac{1}{5!} = \frac{11}{30} \cong 36.6\%$$

$n \to \infty$

$$e^x = \sum_{j=0}^{+\infty} \frac{x^j}{j!}$$

$$\lim_{n \to \infty} p(n) = \sum_{j=0}^{+\infty} \frac{(-1)^j}{j!} = e^{-1} = \frac{1}{e} \cong 0.3679\ldots$$

Ex: Suppose that 10 couples sit on a large round table, taking seats at random. What is the probability that no couple ends up sitting together?



E_i = i^{th} couple sits together $\quad 1 \leq i \leq 10$

$\bigcup_{i=1}^{10} E_i$ ← at least one couple sits together

$\left(\bigcup_{i=1}^{10} E_i\right)^c = \bigcap_{i=1}^{10} E_i^c$ ← no couple sits together

$P\left(\left(\bigcup_{i=1}^{10} E_i\right)^c\right) = 1 - P\left(\bigcup_{i=1}^{10} E_i\right)$ ← desired prob.

$$P\left(\bigcup_{i=1}^{10} E_i\right) = \sum_{k=1}^{10} (-1)^{k+1} \underbrace{\sum_{i_1<\dots<i_k}}_{\binom{10}{k} \text{ terms in this sum.}} \underbrace{P(E_{i_1}\cap\dots\cap E_{i_k})}_{?}$$

$$P(E_{i_1}\cap\dots\cap E_{i_k}) = \boxed{\frac{2^k(19-k)!}{19!}}$$ b/c:

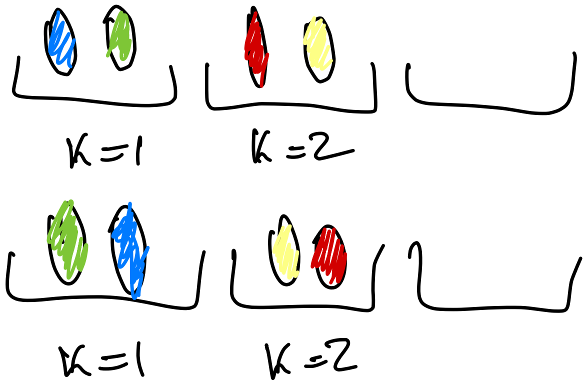
Q1: In how many ways can 20 people sit on round table?

1 2 - - - 20
20 1 2 - - 19
- - -

} shifts need to be identified $\longrightarrow$ $\dfrac{20!}{20} = \boxed{19!}$

Q2: In how many ways can k couples sit together?

"Fix" k couples in their seats, and permute the remaining people. } $(19-k)!$



Permuting seating "within" each couple: still need to multiply by 2^k

so:

A: $2^k(19-k)!$

Thus,

$$P\left(\bigcup_{i=1}^{10} E_i\right) = \sum_{k=1}^{10} (-1)^{k+1} \underbrace{\sum_{i_1<\dots<i_k}}_{\binom{10}{k} \text{ equal terms}} \frac{2^k(19-k)!}{19!}$$

$$= \sum_{k=1}^{10} (-1)^{k+1} \binom{10}{k} \frac{2^k(19-k)!}{19!}$$

$$= \underbrace{\binom{10}{1} 2 \frac{18!}{19!}}_{k=1} - \underbrace{\binom{10}{2} \cdot 2^2 \frac{17!}{19!}}_{k=2} + \dots - \underbrace{\binom{10}{10} 2^{10} \frac{9!}{19!}}_{k=10}$$

$$\cong 0.6605$$

A: Desired probability is $1 - 0.6605 \cong 0.3395$

(33.95%)

EX: If you pick a number at random between 1 and 1000, what is the probability that it is divisible by 7 or by 11?
How about divisible by 7, or 11, or 13?

A = your number is divisible by 7

B = ——————"—————— 11

C = ——————"—————— 13

$P(A \cup B) = ?$ $P(A \cup B \cup C) = ?$

$$P(A) = \frac{142}{1000}, \quad P(B) = \frac{90}{1000}, \quad P(C) = \frac{76}{1000}$$

How many numbers between 1 and 1000 are multiples of 7?

$\lfloor x \rfloor$ = largest integer $\leq x$.

$$\left\lfloor \frac{1000}{7} \right\rfloor = \lfloor 142.857 \rfloor = 142.$$

Similarly, $\left\lfloor \frac{1000}{11} \right\rfloor = \lfloor 90.9091 \rfloor = 90.$

$$\left\lfloor \frac{1000}{13} \right\rfloor = \lfloor 76.92 \rfloor = 76$$

$$P(A \cap B) = \frac{\left\lfloor \frac{1000}{77} \right\rfloor}{1000} = \frac{12}{1000}$$

being divisible by 7 and by 11 $\Longleftrightarrow$ being divisible by 77.

$$P(A\cup B) = P(A) + P(B) - P(A\cap B) = \frac{142}{1000} + \frac{90}{1000} - \frac{12}{1000}$$

$$= \frac{220}{1000} = \boxed{22\,\%}$$

$$P(A\cup B\cup C) = \underbrace{P(A) + P(B) + P(C)}_{k=1} - \underbrace{\big(P(AB) + P(AC) + P(BC)\big)}_{k=2}$$

$$+ \underbrace{P(ABC)}_{k=3}.$$

$$= \frac{142}{1000} + \frac{90}{1000} + \frac{76}{1000} - \left(\frac{12}{1000} + \frac{6}{1000} + \frac{10}{1000}\right)$$

$$= \frac{308}{1000} - \left(\frac{28}{1000}\right)$$

$$= \frac{280}{1000} = \frac{28}{100} = \boxed{28\%}$$

$$P(BC) = \frac{\left\lfloor \frac{1000}{11\cdot 13} \right\rfloor}{1000} = \frac{6}{1000}$$

$$P(AC) = \frac{\left\lfloor \frac{1000}{7\cdot 13} \right\rfloor}{1000} = \frac{10}{1000}$$

$$P(ABC) = 0.$$

$$7\cdot 11\cdot 13 = \underline{\underline{1001}} > 1000.$$