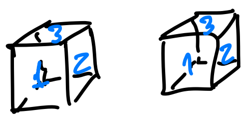
Probability

Def: Sample space S is the collection of all possible outcomes of an experiment. Subsets $E \subset S$ are called events.

Ex: Coin toss $S = \{t, h\}$

Toss 2 dice $S = \{(i,j) : 1 \le i, j \le 6\}$



$$E = \left\{ \begin{array}{lll} (2,2), & (2,4), & (2,6), \\ (4,2), & (4,4), & (4,6), \\ (6,2), & (6,4), & (6,6) \end{array} \right\}$$ "getting even numbers on both dice"

$$F = \{(1,1), (2,2), (3,3), (4,4), (5,5), (6,6)\}$$ "same result on both dice"

Def:
- $E^c = S \setminus E$ complement of E "not E"
- $E \cap F = EF$ intersection of E and F "E and F"
- $E \cup F$ union of E and F "E or F"

Basic properties: Say $E, F, G \subset S$ are events.

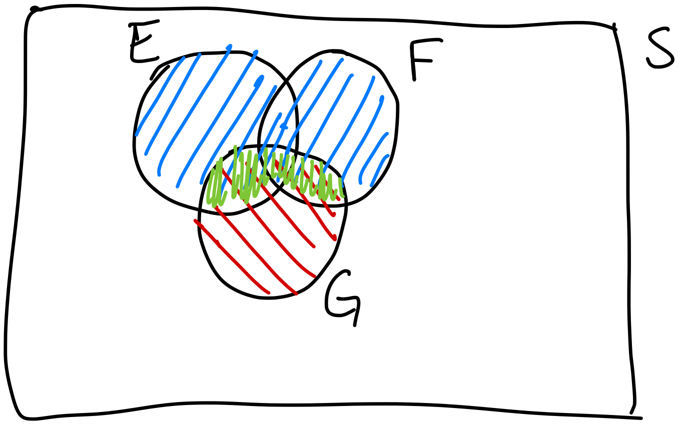
Commutative: $E \cap F = F \cap E$ $(EF = FE)$, $E \cup F = F \cup E$.

Associative: $(E \cap F) \cap G = E \cap (F \cap G) = E \cap F \cap G$

$(E \cup F) \cup G = E \cup (F \cup G) = E \cup F \cup G$

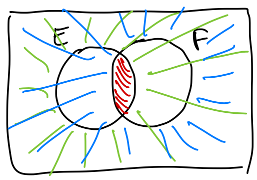
Distributive: $(E \cup F) \cap G = (E \cap G) \cup (F \cap G)$

Diagram:

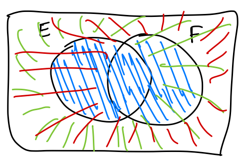


DeMorgan's Laws:

- $(E \cap F)^c = E^c \cup F^c$



- $(E \cup F)^c = E^c \cap F^c$



Say $E_1, E_2, \ldots, E_n$ are events. Then:

- $$\left(\bigcap_{i=1}^{n} E_i \right)^c = \bigcup_{i=1}^{n} E_i^c$$

- $$\left(\bigcup_{i=1}^{n} E_i \right)^c = \bigcap_{i=1}^{n} E_i^c$$

Measure-theoretic probability:

A *probability* on a sample space S is a function

$$P : \underbrace{\mathcal{P}(S)} \longrightarrow [0,1]$$

$\mathcal{P}(S) = \{\text{All subsets of } S\}$
"set of parts of S"

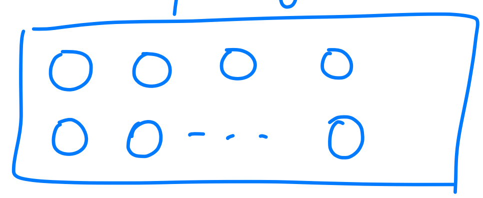
such that

(i) $\forall E \subset S, \quad 0 \leq P(E) \leq 1$

(ii) $P(S) = 1$

(iii) $P\left(\bigcup_{i=1}^{\infty} E_i\right) = \sum_{i=1}^{\infty} P(E_i)$ if $E_i \cap E_j = \phi, \forall i \neq j$

"mutually disjoint events"



"all separate"

Consequences:

- $P(\phi) = 0.$

Pf: $P(\phi \cup S) \overset{(iii)}{=} P(\phi) + \underbrace{P(S)}_{1 \, (ii)}$ b/c $\phi \cap S = \phi$.

$\quad\| \quad S = S \cup \phi.$

$P(S) \overset{(ii)}{=} 1$

$$P(\phi) + 1 = 1$$
$$P(\phi) = 0.$$

- Probabilities when all outcomes in sample space are equally likely.

$S = \{e_1, e_2, \ldots, e_n\} \leftarrow$ finite set: $|S| = n$

All outcomes are equally likely: $P(\{e_i\}) = P(\{e_j\}) \ \forall i,j$

Since they are all equal, let $p = P(\{e_i\}) \in [0,1]$.

$$1 \overset{(ii)}{=} P(S) \overset{(iii)}{=} \underbrace{P(\{e_1\})}_{p} + \underbrace{P(\{e_2\})}_{p} + \cdots + \underbrace{P(\{e_n\})}_{p} = n \cdot p$$

$$\Longrightarrow \quad p = \frac{1}{n} = \frac{1}{|S|}.$$

Given any event $E \subset S$, it can be written as:

$$E = \{e_{i_1}, e_{i_2}, \ldots, e_{i_k}\} \qquad 0 \le k \le n.$$

$$P(E) = \underbrace{P(\{e_{i_1}\})}_{p} + \underbrace{P(\{e_{i_2}\})}_{p} + \cdots + \underbrace{P(\{e_{i_k}\})}_{p} = k \cdot p = \frac{k}{n}$$

$$P(E) = \frac{k}{n} = \frac{|E|}{|S|}$$

<u>Upshot</u>: When all outcomes are ~~equally likely~~, and the sample space S is finite $(|S| < \infty)$, then

$$P(E) = \frac{|E|}{|S|} = \frac{\#\text{ of elements of } E}{\#\text{ of elements of } S}$$

<u>Ex</u>:
- Coin toss: $S = \{t, h\}$ $\quad P(t) = \frac{1}{2} = \frac{\#\{t\}}{|S|}$
- Throwing 2 dice $\quad S = \{(i,j) : 1 \le i,j \le 6\}$.

$|S| = 6 \cdot 6 = \underline{\underline{36}}$

$|E| = 9$, $|F| = 6$

Getting even #s on both dice

Getting same # on both dice

$P(E) = \frac{|E|}{|S|} = \frac{9}{36} = \boxed{\frac{1}{4}}$

$P(F) = \frac{|F|}{|S|} = \frac{6}{36} = \boxed{\frac{1}{6}}$

Cards: 52 cards, 4 suits w/ 13 denominations

A, 2, 3, 4, 5, 6, 7, 8, 9, 10, J, Q, K

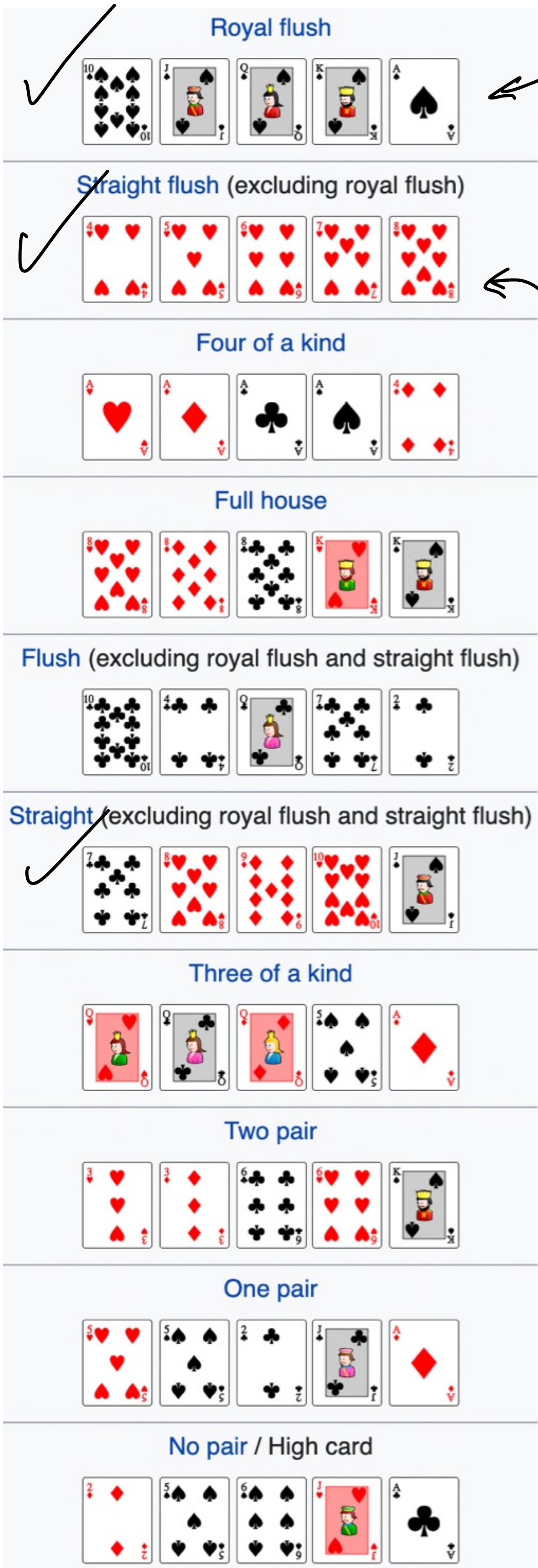
Poker: Hand = (5 cards)

$S = \{\text{set of all hands}\}$ ← sample space.

Q: How many hands of poker are possible?

$|S| = ?$

$$\binom{52}{5} = \frac{52!}{47!\,5!} = 2{,}598{,}960.$$



$(10, J, Q, K, A)$ any suit

Q: How many Royal flushes are there?

A: 4

$$P(\text{Royal flush}) = \frac{4}{\binom{52}{5}}$$

Straight: All cards are consecutive

Flush: All of the same suit.

Q: How many straight flush hands?

A:

$$9\begin{cases} A\ 2\ 3\ 4\ 5 \\ 2\ 3\ 4\ 5\ 6 \\ \vdots \\ 9\ 10\ J\ Q\ K \end{cases} \times 4 \text{ suits} = \underline{\underline{36}}$$

$\boxed{\cancel{10\ J\ Q\ K\ A}}$ ← Royal flush

$$P(\text{straight flush}) = \frac{36}{\binom{52}{5}}$$

Q: How many straight hands?

A 2 3 4 5 of any suit $\quad 4 \cdot 4 \cdot 4 \cdot 4 \cdot 4 = 4^5$

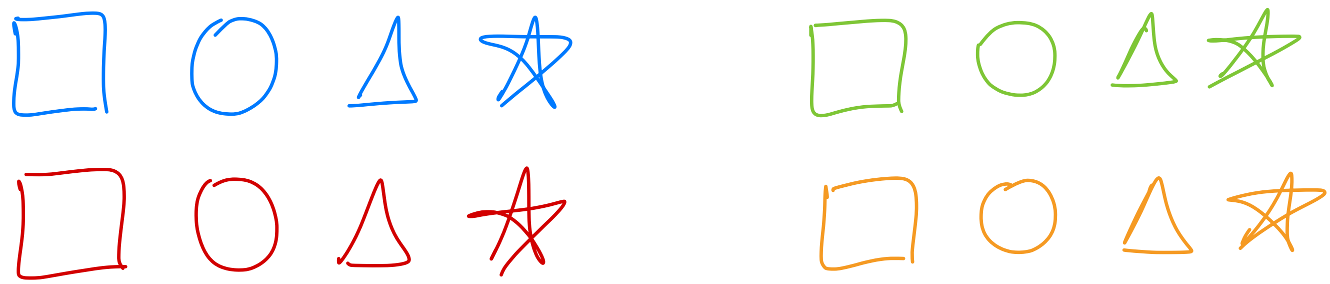
— of same suit $\quad 4 \cdot 1 \cdot 1 \cdot 1 \cdot 1 = 4$

A 2 3 4 5 not all of same suit $\quad 4^5 - 4$

A: $10 \cdot (4^5 - 4)$ straight hands.

$$P(\text{Straight}) = \frac{10(4^5-4)}{\binom{52}{5}} \cong 0.0039 \qquad (0.39\%)$$

Ex: Suppose a children's game consist of 16 pieces of 4 colors:



All pieces are given to 2 players. How many sets of pieces can a player receive?

$$\binom{16}{8} = \frac{16!}{8!\,8!}$$

What is the probability that both players receive the same number of blue pieces?

$$\frac{\binom{4}{2}\binom{12}{6}}{\binom{16}{8}} = \frac{\frac{4!}{\cancel{2!2!}}\,\frac{12!}{\cancel{6!6!}}}{\frac{16!}{\cancel{8!8!}}} = \frac{\cancel{12!}\,4!\,\cancel{4}^{2}\cdot 7^{2}}{\cancel{16}\cdot 15\cdot \cancel{14}\cdot 13\cdot \cancel{12!}}$$

$$= \frac{7\cdot\cancel{12}\,4}{13\cdot\cancel{15}\,5} = \frac{28}{65} \cong 43.07\%$$

What is the probability that one player receives all the squares?

$$\frac{\binom{4}{4}\cdot\binom{12}{4}}{\binom{16}{8}} = \frac{1}{26} \approx 3.8\,\%$$