MAT330/681 Lecture 27 5/12/2021

Review (Part I)

Info about Final Exam:

- Available on Blackboard on May 19, from 12:00 am to 11:59 pm
- Multiple attempts allowed, final score is average of attempts, 2 hours total.
- Questions are very similar to those done in lectures and homework assignments.

Topics to Review

- HW3 #3 ✓
- HW8 #1, 2, 3 ✓
- HW7 #2 ✓
- HW 12 #2 ✓

HW3 #3:

20% migrate south
10% need 800 cal/day & migrate south

M = migrate south

C = need to consume at least 800 cal/day to survive.

$P(M) = \frac{2}{10}$, $P(C) = ?$ $P(MC) = \frac{1}{10}$

Q: Prob. bird does not need 800 cal given that they migrate?

$$P(C^c|M) = 1 - P(C|M) = 1 - \frac{P(CM)}{P(M)} = 1 - \frac{1/\cancel{10}}{2/\cancel{10}} = \frac{1}{2}.$$

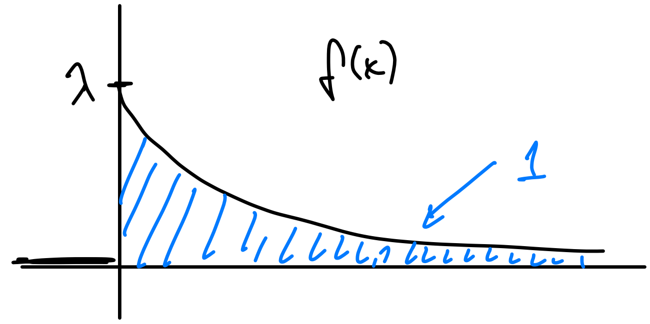
$P(\cdot|E)$ is a probability, so it satisfies

$0 \le P(\cdot|E) \le 1$, $P(A^c|E) = 1 - P(A|E)$, ...

$$P(A|B) = \frac{P(AB)}{P(B)}$$

HW8

$X \sim \text{Exponential}(\lambda)$



p.d.f. $f(x) = \begin{cases} \lambda e^{-\lambda x}, & \text{if } x \geq 0 \\ 0, & \text{if } x < 0 \end{cases}$

c.d.f. $F(x) = \int_{-\infty}^{x} f(t)\,dt = \int_0^x \lambda e^{-\lambda t}\,dt = \lambda \left(-\frac{e^{-\lambda t}}{\lambda} \right)\Big|_0^x$

$= -e^{-\lambda x} + 1 = 1 - e^{-\lambda x}.$

$F(x) = P(X \leq x) = 1 - e^{-\lambda x}$

$P(X \geq x) = 1 - F(x) = 1 - (1 - e^{-\lambda x}) = e^{-\lambda x}$

#1 X = # years that tree lives

$X \sim \text{Exponential}(\lambda) \Rightarrow E(X) = \frac{1}{\lambda} = 183 \Rightarrow \lambda = \frac{1}{183}$

$P(X \geq 100) = e^{-\frac{1}{183} \cdot 100} = e^{-\frac{100}{183}}$

#2 Hazard rate:

$\lambda(t) = \frac{f(t)}{1 - F(t)} \underset{\downarrow}{\overset{X \sim \text{Exponential}(\lambda)}{=}} \frac{\lambda \cancel{e^{-\lambda t}}}{\cancel{e^{-\lambda t}}} = \lambda.$

If $\lambda(t) = 6t + 4$ is the Hazard rate of X,

find $P(X < \frac{1}{2}) = F\left(\frac{1}{2}\right)$ $\qquad F'(s) = f(s)$

$\rightsquigarrow$ Knowing $\lambda(t)$ implies knowing $F(t)$.

$$\int_0^t \lambda(s)\,ds = \int_0^t \frac{f(s)}{1-F(s)}\,ds = -\int_0^t \frac{\overbrace{-f(s)}^{=\frac{d}{ds}(1-F(s))}}{1-F(s)}\,ds$$

$$\int \frac{\phi'(s)}{\phi(s)}\,ds = \log|\phi(s)|$$

$$= -\log(1-F(s))\Big|_0^t = -\log(1-F(t)) + \cancelto{0}{\log(1)}$$

$$F(0)=0$$

$$\boxed{\int_0^t \lambda(s)\,ds = -\log(1-F(t))}$$

Solve for $F(t)$:

$$e^{\int_0^t \lambda(s)\,ds} = e^{-\log(1-F(t))} = e^{\log\left(\frac{1}{1-F(t)}\right)} = \frac{1}{1-F(t)}$$

$$1-F(t) = e^{-\int_0^t \lambda(s)\,ds}$$

$$-F(t) = -1 + e^{-\int_0^t \lambda(s)\,ds}$$

$$\boxed{F(t) = 1 - e^{-\int_0^t \lambda(s)\,ds}}$$

output: cdf

input λ (Hazard rate)

if $\lambda(t) = 6t+4$, then

$$\int_0^t \lambda(s)\,ds = \int_0^t 6s+4\,ds$$

$$= (3s^2+4s)\Big|_0^t$$

$$= 3t^2+4t$$

$$F(t) = 1 - e^{-(3t^2+4t)}$$

$$P\left(X \le \tfrac{1}{2}\right) = F\left(\tfrac{1}{2}\right) = 1 - e^{-\left(\frac{3}{4}+2\right)} = \underline{\underline{1 - e^{-\frac{11}{4}}}}$$

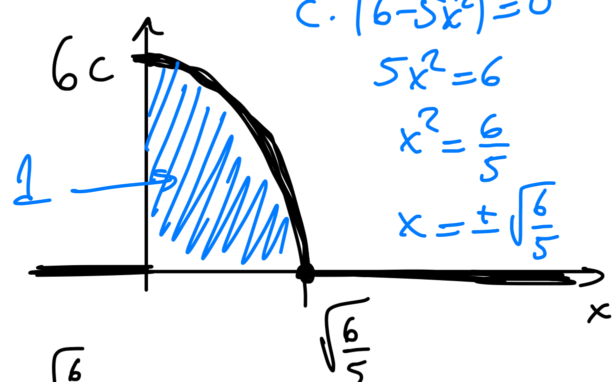
<u>#3</u> Continuous Random Variables

X is positive (cont.) random variable.

$$f(x) = \begin{cases} c\cdot(6-5x^2), & \text{if } 0 \le x \le \sqrt{\frac{6}{5}} \\ 0, & \text{if } x < 0 \end{cases} \qquad c \in \mathbb{R}$$

1. $c \in \mathbb{R}$ ✓
2. $E(X)$ ✓
3. $E(X^2)$ ✓
4. $Var(X) = E(X^2) - E(X)^2$

$$\int_{-\infty}^{+\infty} f(x)\,dx = 1, \quad f(x) \ge 0$$



$c\cdot(6-5x^2) = 0$

$5x^2 = 6$

$x^2 = \frac{6}{5}$

$x = \pm\sqrt{\frac{6}{5}}$

$$1 = \int_0^{\sqrt{\frac{6}{5}}} c\cdot(6-5x^2)\,dx = c\left(6x - \frac{5x^3}{3}\right)\Bigg|_0^{\sqrt{\frac{6}{5}}} = c\underbrace{\left(6\sqrt{\frac{6}{5}} - \frac{\cancel{5}}{\cancel{3}}\,\frac{\cancel{6}^{\,2}}{\cancel{5}}\sqrt{\frac{6}{5}}\right)}_{4\sqrt{\frac{6}{5}}}$$

$$c = \frac{1}{4\sqrt{\frac{6}{5}}} = \frac{\sqrt{5}}{4\sqrt{6}}$$

$$E(X) = \int_0^{\sqrt{6/5}} x f(x)\, dx = \frac{\sqrt{5}}{4\sqrt{6}} \int_0^{\sqrt{6/5}} x(6-5x^2)\, dx =$$

$$= \frac{\sqrt{5}}{4\sqrt{6}} \left(3x^2 - \frac{5x^4}{4} \right) \Bigg|_0^{\sqrt{6/5}} = \frac{\sqrt{5}}{4\sqrt{6}} \left(3\,\frac{6}{5} - \frac{\cancel{5}}{\cancel{4}} \cdot \frac{\cancel{36}^{\,9}}{\cancel{25}_{\,5}} \right)$$

$$= \frac{\sqrt{5}}{4\sqrt{6 \cdot 5}} \cdot 9 = \frac{9}{4\sqrt{30}}$$

$$E(X^2) = \int_0^{\sqrt{\frac{6}{5}}} x^2 f(x)\, dx = \frac{\sqrt{5}}{4\sqrt{6}} \int_0^{\sqrt{\frac{6}{5}}} x^2(6-5x^2)\, dx = \cdots = \frac{6}{25}$$

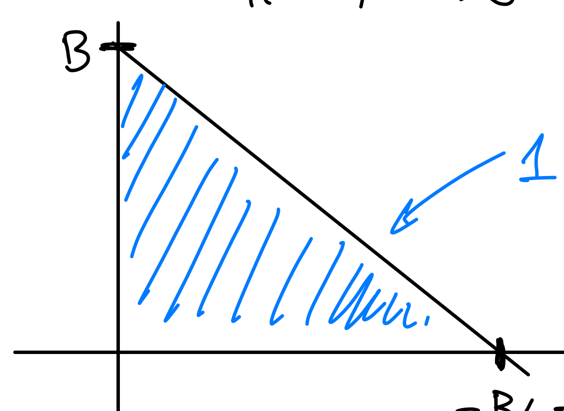
$$E(g(X)) = \int g(x) f(x)\, dx$$

$$Var(X) = E(X^2) - E(X)^2 = \frac{6}{25} - \left(\frac{9}{4\sqrt{30}} \right)^2 = \frac{57}{800} \cong 0.07125$$

HW7 #2

$$f(x) = \begin{cases} Ax + B, & \text{if } x \in [0, 8] \\ 0, & \text{otherwise} \end{cases}$$

$A < 0, \quad B > 0$



$E(X) = ?$

$Var(X) = ?$

$$1 = \text{Area} = \frac{1}{2}\left(-\frac{B}{A}\right) \cdot B = -\frac{B^2}{2A}$$

$$\boxed{2A = -B^2}$$

$Ax + B = 0$

$x = -\frac{B}{A}$

$-\frac{B}{A}=8$ $\quad A=-\frac{B^2}{2}=-\frac{B}{8} \Rightarrow B^2=\frac{B}{4} \Rightarrow B\left(B-\frac{1}{4}\right)=0$

$\Rightarrow \boxed{B=\frac{1}{4}}$

$\Rightarrow A=-\frac{1}{4}\cdot\frac{1}{8}$

$\Rightarrow \boxed{A=-\frac{1}{32}}$

$$E(X)=\int_0^8 x f(x)\,dx$$

$$=\int_0^8 x(Ax+B)\,dx=\int_0^8 x\left(-\frac{1}{32}x+\frac{1}{4}\right)dx$$

$$=-\frac{1}{32}\,\frac{x^3}{3}\Big|_0^8+\frac{x^2}{8}\Big|_0^8=-\frac{1}{32}\left(\frac{512}{3}\right)+8=-\frac{16}{3}+8$$

$$=\frac{24-16}{3}=\boxed{\frac{8}{3}}$$

HW 12 #2

$n=25$ dice $\quad X_1, X_2, \ldots, X_{25}$

$$\overline{X_{25}}=\frac{X_1+\cdots+X_{25}}{25}$$

$\frac{A+B}{25}=7 \Rightarrow \frac{48+B}{25}=7 \Rightarrow 48+B=175$

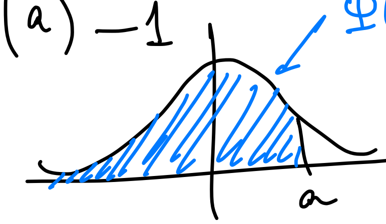
$\Rightarrow \boxed{B=127}$ CLT

$\Phi(z)=P(Z\le a)$

$Z\sim \text{Normal}(0,1)$

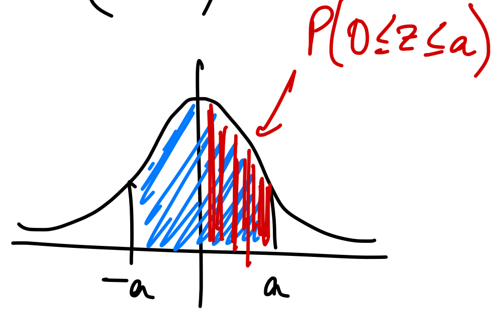
$$P(48\le X_1+\cdots+X_{25}\le 127)\cong 2\Phi(a)-1$$

$\Phi(a)$



Q: Find a

CLT: $\dfrac{\overline{X}_n-\mu}{\sigma/\sqrt{n}} \xrightarrow{\;n\to\infty\;} Z \sim \text{Normal}(0,1)$



Approx. works by:

$$P\left(-a \le \frac{\overline{X}_n-\mu}{\sigma/\sqrt{n}} \le a\right) = P(-a \le Z \le a) = 2P(0 \le Z \le a)$$
$$= 2\Phi(a) - 1.$$

Find these values!

$$P\left(\frac{48}{25} \le \underbrace{\frac{X_1+\dots+X_{25}}{25}}_{=\overline{X}_{25}} \le \frac{127}{25}\right) = P\left(\frac{48}{25}-\frac{7}{2} \le \overline{X}_{25}-\mu \le \frac{127}{25}-\frac{7}{2}\right) = \dots$$

$$\mu = E(X) = \frac{7}{2}, \quad Var(X) = \frac{35}{12} \Rightarrow \sigma = \sqrt{\frac{35}{12}}$$

$$\Rightarrow \frac{\sigma}{\sqrt{n}} = \sqrt{\frac{35}{12}} \cdot \frac{1}{5}$$

$$\dots = P\left(\underbrace{\frac{\overbrace{\frac{48}{25}-\frac{7}{2}}^{=-\frac{79}{50}}}{\frac{1}{5}\sqrt{\frac{35}{12}}}}_{=-a} \le \frac{\overline{X}_{25}-\mu}{\sigma/\sqrt{n}} \le \underbrace{\frac{\overbrace{\frac{127}{25}-\frac{7}{2}}^{=\frac{79}{50}}}{\frac{1}{5}\sqrt{\frac{35}{12}}}}_{=a}\right) = 2\Phi(a) - 1$$

$$a = \frac{79}{\cancel{50}_{10}} \Big/ \frac{1}{\cancel{5}}\sqrt{\frac{35}{12}} = \frac{79}{10}\sqrt{\frac{12}{35}} \simeq 4.63$$