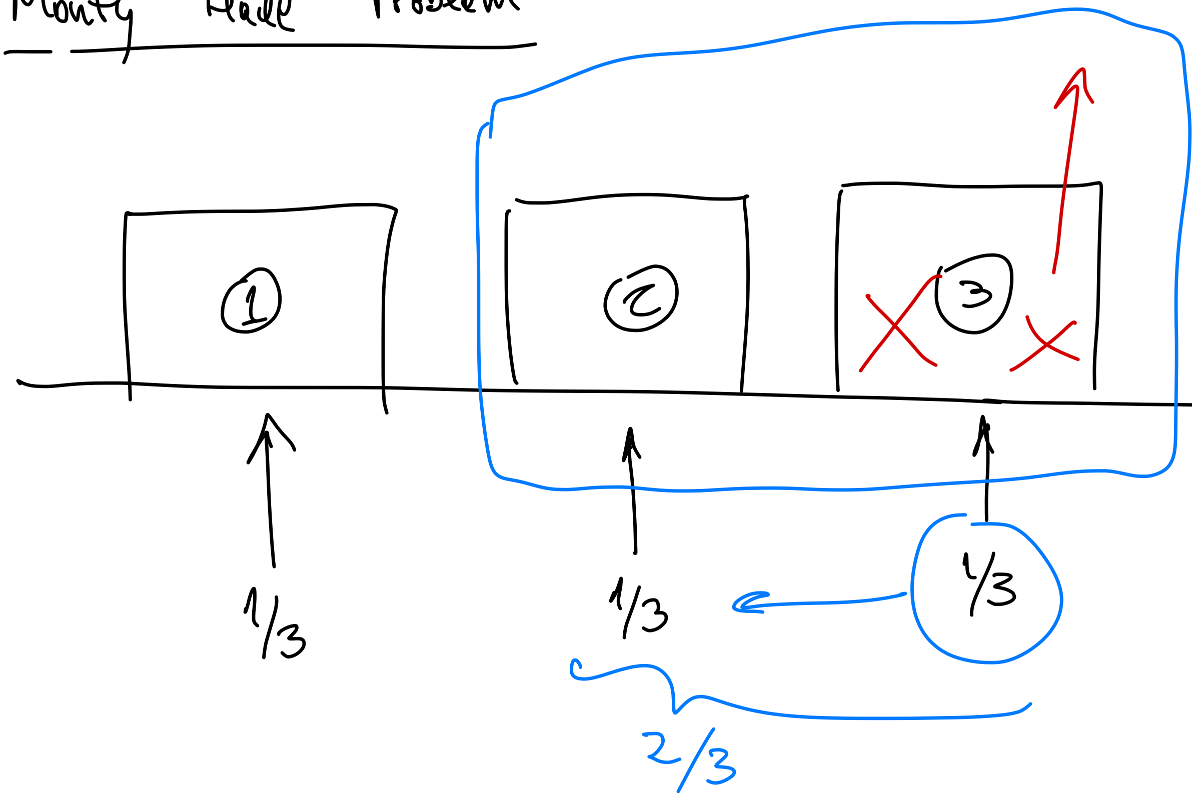
MAT330/681 Lecture 1 2/1/2021

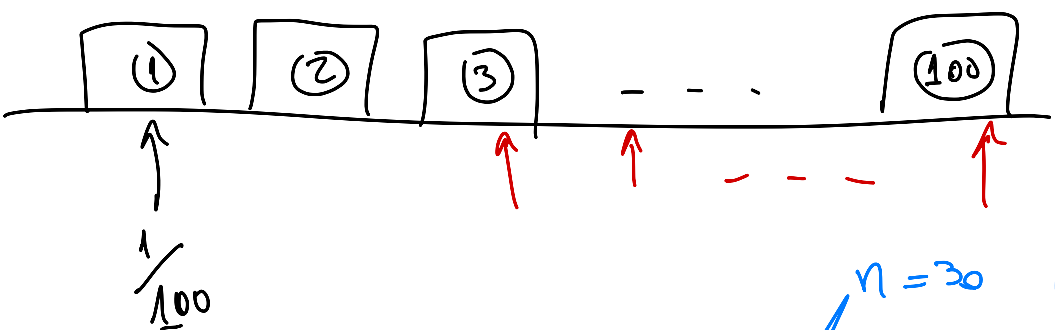
Monty Hall Problem



→ Bayes' Thm

→ conditional probability

Better intuition:



$n = 30$ $(n \to \infty)$

Matching problem: Suppose 30 students are in a classroom and have identical phones. The professor collects all the phones from students and places them in a common bucket. At the end of class, each student picks one phone from the bucket. What is the probability that none of the students picks their own phone?

Answer: $p = \sum_{j=0}^{n} \frac{(-1)^j}{j!} \approx 36.7\%$ as $n \to \infty$

(Inclusion-exclusion principle)

Basic counting:

Q: Lehman's cafeteria has the following lunch options

3 options of sides	2 options for protein	5 options of drink

How many possible meals are there in total?

$$\underbrace{3}\cdot\underbrace{2}\cdot\underbrace{5} = \underline{30 \text{ choices}}.$$

More generally: "Basic Principle of Counting"

Experiment 1: n_1 outcomes
Experiment 2: n_2 outcomes
⋮
Experiment k: n_k outcomes

Collection of outcomes for all experiments has a total of

$n_1 \cdot n_2 \cdot \ldots \cdot n_k$

possible outcomes.

Ex: License plates

$$\underbrace{A}_{\text{letter}}\ \underbrace{B}_{\text{letter}}\ \underbrace{C}_{\text{letter}}\ \underbrace{2}_{\#}\ \underbrace{0}_{\#}\ \underbrace{2}_{\#}\ \underbrace{1}_{\#}$$

Experiment 1, –, 3: 26 letters
Experiments 4, – 7: 10 numbers

total: $26^3 \cdot 10^4 = 175{,}760{,}000$

Permutations

Factorial: $n! = n(n-1)(n-2)\cdots 1$, $n \in \mathbb{N}$ ($0! = 1$ by convention)

There are $n!$ ways of reshuffling n objects.

- ABC $n=3$ objects (distinct)

ABC	BAC	CAB
ACB	BCA	CBA

$3.2.1$

$3! = 6$ permutations

- MATH $n=4$ objects (distinct) $\rightsquigarrow$ $4! = 4.3.2.1 = 24$ permutations

- MATT $n=4$ objects (2 of them are alike)

$$\frac{4!}{2!} = \frac{24}{2} = 12 \text{ permutations}$$

TMAT
TMAT
double counted

- PEPPER $n=6$ objects
(3 of them are alike — P)
(2 of them are alike — E)

$$\frac{6!}{3!\,2!} = \frac{6.5.4.\cancel{3!}}{\cancel{3!}\,2!} = \frac{6.5.\cancel{4}^{2}}{\cancel{2}} = \boxed{60}$$

$P_1\ P_2\ P_3$ $E_1\ E_2$

$$\begin{pmatrix} P_1 E P_2 P_3 R E \\ P_3 E P_1 P_2 R E \\ \vdots \end{pmatrix} \rightsquigarrow \text{PEPPRE}$$

$3! = 6$ redundancy is the P's

$$\begin{pmatrix} P E_1 P P R E_2 \\ P E_2 P P R E_1 \end{pmatrix} \rightsquigarrow \text{PEPPRE}$$

$2! = 2$ redundancy in the E's

n objects

n_1 of them are alike

n_2 of them are alike

$\vdots$

n_k of them are alike

Total number of permutations:

$$\frac{n!}{n_1!\, n_2! \cdots n_k!}$$

ABRAKATABRA

$$\frac{11!}{5!\,2!\,2!}$$

~~$11! = 39,916,800$~~ instead:

$$11! = 11 \cdot 10 \cdot 9 \cdot 8 \cdot 7 \cdot 6 \cdot 5!$$

$$\frac{11!}{5!\,2!\,2!} = \frac{11 \cdot 10 \cdot 9 \cdot \cancel{8}^{2} \cdot 7 \cdot 6}{\cancel{4}} = 83,160$$

Q: In how many ways can the letters in the word OPTICAL be permuted such that all the vowels are kept together?

$\boxed{\text{OIA}}$PTCL

T$\boxed{\text{AIO}}$PCL

treat this as a "combo-letter"

$n = 5$ objects: OIA, P, T, C, L

$n' = 3$ objects within the combo-letter O, I, A

A: $5! \cdot 3! = 120 \cdot 6 = \boxed{720}$

Combinations:

Q: From n objects, we want to choose k in no particular order.

If the order matters

$$\underbrace{n}_{1^{st}\text{ choice}} \cdot \underbrace{(n-1)}_{2^{nd}\text{ choice}} \underbrace{(n-2)}_{3^{rd}\text{ choice}} \cdots \underbrace{n-(k-1)}_{k^{th}\text{ choice}} = (n-k+1)$$

If the order does not matter

$$\frac{n(n-1)(n-2)\cdots(n-k+1)}{k!}$$

permutations of the order of choices.

$$\frac{n(n-1)(n-2)\cdots(n-k+1)}{k!} = \frac{n(n-1)(n-2)\cdots(n-k+1)\overbrace{(n-k)(n-k-1)\cdots 1}^{(n-k)!}}{k! \cdot \underbrace{(n-k)(n-k-1)\cdots 1}_{(n-k)!}}$$

$$= \frac{n!}{k!(n-k)!}$$

say: "n choose k"

Def: $\binom{n}{k} = \frac{n!}{k!(n-k)!}$ is called a binomial coefficient

Q: Suppose you have 10 blueberries and 7 raspberries in a bowl. In how many ways can you eat these 17 berries one by one?

$$\underbrace{R}_{1}\ \ \underbrace{B}_{2}\ \ \underbrace{R}_{3}\ \ ---\ \ \underbrace{B}_{17}$$

Choose 7 of the 17 "slots" to place a R

$$\binom{17}{7} = \frac{17!}{7!\,10!} = \binom{17}{10}$$