Recall:
$$\liminf_{n\to\infty} s_n = \lim_{N\to\infty} \inf\{s_n : n>N\}$$
$$\limsup_{n\to\infty} s_n = \lim_{N\to\infty} \sup\{s_n : n>N\}$$

Properties:

- $\liminf_{n\to\infty} s_n \leq \limsup_{n\to\infty} s_n$

  $= \iff s_n \to L = \liminf_{n\to\infty} s_n = \limsup_{n\to\infty} s_n$

- $\limsup_{n\to\infty} s_n \cdot t_n = s \cdot \limsup_{n\to\infty} t_n$

  if $s_n \to s$ (including if $\limsup_{n\to\infty} t_n = \pm\infty$)

Theorem. Let $(s_n)_{n\in\mathbb{N}}$ be a sequence such that $s_n \neq 0$ for all $n\in\mathbb{N}$. Then
$$\liminf_{n\to\infty}\left|\frac{s_{n+1}}{s_n}\right| \leq \liminf_{n\to\infty}|s_n|^{1/n} \leq \limsup_{n\to\infty}|s_n|^{1/n} \leq \limsup_{n\to\infty}\left|\frac{s_{n+1}}{s_n}\right|.$$

Pf: Focus on the last inequality (the others are trivial or analogous).
$$\underbrace{\limsup_{n\to\infty}|s_n|^{1/n}}_{=:\alpha} \leq \underbrace{\limsup_{n\to\infty}\left|\frac{s_{n+1}}{s_n}\right|}_{=:L}.$$

There is nothing to be done if $L=+\infty$; so assume $L<\infty$.

Proving $\alpha \le L$ is equivalent to proving that

$$\alpha \le L_1 \quad \text{for all} \quad L_1 > L.$$

$$L = \limsup_{n\to\infty} \left|\frac{s_{n+1}}{s_n}\right| = \lim_{N\to\infty} \sup\left\{ \left|\frac{s_{n+1}}{s_n}\right| : n > N \right\} < L_1$$

Thus $\exists N \in \mathbb{N}$, such that $\sup\left\{ \left|\frac{s_{n+1}}{s_n}\right| : n > N \right\} < L_1$.

Note that $\left|\frac{s_{n+1}}{s_n}\right| < L_1$ for all $n > N$.

$$|s_n| = \underbrace{\left|\frac{s_n}{s_{n-1}}\right| \left|\frac{s_{n-1}}{s_{n-2}}\right| \left|\frac{s_{n-2}}{s_{n-3}}\right| \cdots \left|\frac{s_{N+1}}{s_N}\right|}_{n-(N+1)+1 = n-N} |s_N|$$

so $|s_n| < L_1^{n-N} \cdot |s_N| = L_1^{n} \cdot a$,

where $a = \frac{|s_N|}{L^N}$. Taking n^{th} roots of both sides:

$$|s_n|^{\frac{1}{n}} < L_1 a^{1/n} \quad \text{for all } n > N$$

Since $\lim_{n\to\infty} a^{\frac{1}{n}} = 1$, $\underbrace{\limsup_{n\to\infty} |s_n|^{\frac{1}{n}}}_{\alpha} \le L_1 \cdot \underbrace{\lim_{n\to\infty} a^{1/n}}_{=1} \le L_1$

Thus $\alpha \le L_1$ (for all $L_1 > L$), so $\alpha \le L$ as desired.

$\square$

<u>Cor:</u> If $\left|\frac{s_{n+1}}{s_n}\right| \to L$, then $|s_n|^{\frac{1}{n}} \to L$.

<u>Series.</u>

$$\sum_{n=1}^{+\infty} a_n = a_1 + a_2 + a_3 + \dots + a_n + \dots$$ an infinite sum.

Sequence of partial sums of the series $\sum_{n=1}^{+\infty} a_n$:

$$s_1 = a_1$$

$$s_2 = a_1 + a_2$$

$$s_3 = a_1 + a_2 + a_3$$

$$\vdots$$

$$s_n = \underbrace{a_1 + a_2 + a_3 + \dots + a_n}_{\text{sum of first } n \text{ terms of the series } \sum_{n=1}^{\infty} a_n.} = \sum_{k=1}^{n} a_k$$

<u>Def:</u> The series $\sum_{n=1}^{\infty} a_n$ <u>converges</u> to L if and only if

the sequence $s_n := \sum_{k=1}^{n} a_k$ converges to L.

(i.e. $\forall \varepsilon > 0\ \exists N \in \mathbb{N},\ n \ge N \Rightarrow |s_n - L| < \varepsilon$.)

Moreover, $\sum_{n=1}^{\infty} a_n$ converges absolutely if the series $\sum_{n=1}^{\infty} |a_n|$ converges.

Geometric Series.

$$\sum_{n=0}^{\infty} a_n = \sum_{n=0}^{+\infty} a_0 \cdot r^n \qquad a_{n+1} = a_n \cdot r \quad \text{for some } \overset{\boxed{\text{ratio}}}{r} \in \mathbb{R}$$

for all $n \in \mathbb{N} \cup \{0\}$.

WLOG: $a_0 \neq 0$.

First term: a_0

2^{nd} term: $a_1 = a_0 \cdot r$

3^{rd} term: $a_2 = a_1 \cdot r = a_0 \cdot r^2$

4^{th} term: $a_3 = a_2 \cdot r = a_1 \cdot r^2 = a_0 \cdot r^3$

$\vdots$

$(n+1)^{th}$ term: $a_n = \cdots = a_0 \cdot r^n$.

Theorem: $\sum_{n=0}^{+\infty} a_0 \cdot r^n$ converges if and only if $|r| < 1$, in which case,

$$\sum_{n=0}^{+\infty} a_0 \cdot r^n = \frac{a_0}{1-r}$$

Pf: Partial sums of $\sum_{n=0}^{+\infty} a_0 \cdot r^n$ are:

$$S_n = \sum_{k=0}^{n} a_0 r^k = a_0 + a_0 \cdot r + a_0 \cdot r^2 + \cdots + a_0 \cdot r^n$$

$$S_n = a_0 + \cancel{a_0 r} + \cancel{a_0 r^2} + \cancel{\cdots} + \cancel{a_0 r^n}$$

$$- S_n \cdot r = \quad - \cancel{a_0 \cdot r} - \cancel{a_0 \cdot r^2} - \cancel{a_0 r^3} - \cancel{\cdots} - a_0 \cdot r^{n+1}$$

$$S_n - S_n \cdot r = a_0 - a_0 \cdot r^{n+1}$$

$$S_n (1 - r) = a_0 (1 - r^{n+1})$$

If $r \neq 1$, then:

$$S_n = a_0 \frac{1 - r^{n+1}}{1 - r} \quad \text{for all } n \in \mathbb{N}.$$

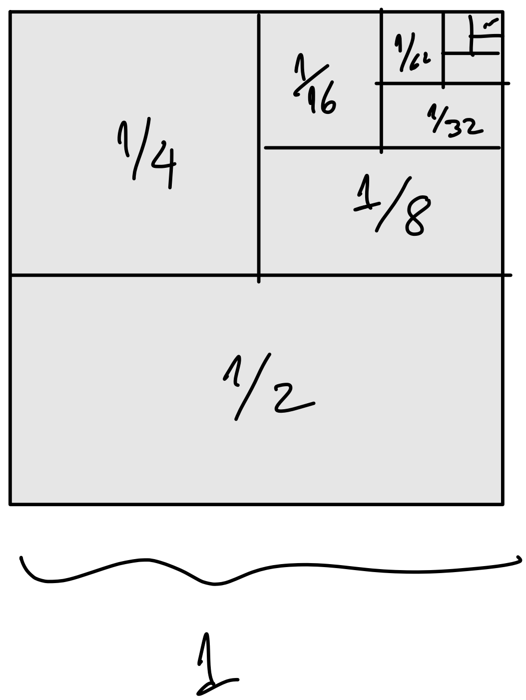
The sequence $S_n = a_0 \dfrac{1 - r^{n+1}}{1 - r}$ converges if and only if $r \neq 1$ and $\lim\limits_{n \to \infty} r^{n+1}$ exists, which holds if and only if $|r| < 1$, in which case $\lim\limits_{n \to \infty} r^{n+1} = 0$:

$$\sum_{n=0}^{+\infty} a_0 \cdot r^n = \lim_{n \to \infty} a_0 \frac{1 - r^{n+1}}{1 - r} = \frac{a_0}{1 - r}.$$

0

If $|r| \geq 1$, then $\sum\limits_{n=0}^{+\infty} a_0 \cdot r^n$ diverges, b/c so does the sequence of partial sums. $\square$

Why is it called "geometric" series?



$$\text{Area} = 1$$

$$= \frac{1}{2} + \frac{1}{2}$$

$$= \frac{1}{2} + \frac{1}{4} + \frac{1}{4}$$

$$= \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{8}$$

$$\cdots = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \frac{1}{32} + \cdots$$

$$= \sum_{n=0}^{+\infty} \frac{1}{2} \cdot \left(\frac{1}{2}\right)^n$$

$$\sum_{n=0}^{+\infty} \underbrace{\frac{1}{2}}_{a_0} \cdot \Big(\underbrace{\frac{1}{2}}_{r}\Big)^n = \frac{a_0}{1-r} = \frac{1/2}{1-1/2} = \frac{1/2}{1/2} = 1.$$

$$a_0 = r = \frac{1}{2}$$

Theorem (Cauchy Criterion). The series $\sum_{n=1}^{\infty} a_n$ converges if and only if $\forall \varepsilon > 0 \ \exists N \in \mathbb{N}$ such that

$$n \geq m > N \implies \left| \sum_{k=m+1}^{n} a_k \right| < \varepsilon$$

$= a_{m+1} + \dots + a_n$

Pf: Consider the sequence of partial sums, $s_n = \sum_{k=1}^{n} a_k$. This sequence $(s_n)_{n\in\mathbb{N}}$ is Cauchy if and only if $\forall \varepsilon > 0 \ \exists N \in \mathbb{N}$ s.t.

$$n \geq m > N \implies |s_n - s_m| < \varepsilon$$

$$\left.\begin{aligned} s_n &= \sum_{k=1}^{n} a_k = a_1 + a_2 + \dots + a_m + \dots + a_n \\ s_m &= \sum_{k=1}^{m} a_k = a_1 + a_2 + \dots + a_m \\ s_n - s_m &= a_{m+1} + a_{m+2} + \dots + a_n = \sum_{k=m+1}^{n} a_k \end{aligned}\right\} \quad |s_n - s_m| = \left| \sum_{k=m+1}^{n} a_k \right| < \varepsilon.$$

The sequence $(s_n)_{n\in\mathbb{N}}$ is Cauchy if and only if it converges.

$\square$

Corollary ("n^{th} term test"). If $\sum_{n=1}^{+\infty} a_n$ converges, then

$$\lim_{n\to\infty} a_n = 0.$$

Pf: Apply Thm (Cauchy criterion) w/ $n = m+1$:

$\sum_{n=1}^{\infty} a_n$ converges $\Longrightarrow$ $\forall \varepsilon > 0 \;\exists N \in \mathbb{N}$

$$n \geq N \Rightarrow \Big| \underbrace{\sum_{k=n}^{n} a_k}_{= a_n} \Big| < \varepsilon \quad \text{i.e. } |a_n| < \varepsilon.$$

i.e. $\lim_{n \to \infty} a_n = 0$. $\square$

Example: $$\sum_{n=1}^{\infty} \frac{n^2+1}{n^2+n+7} \text{ does not converge.}$$

b/c $a_n = \dfrac{n^2+1}{n^2+n+7}$ has limit $\lim_{n \to \infty} a_n = 1 \neq 0$.

⚠ **Warning**: "Converse" to n^{th} term test fails:

the harmonic series

$$\sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \dots$$

diverges, but $\lim_{n \to \infty} \frac{1}{n} = 0$.

Comparison Test.

Let $\sum_{n=1}^{\infty} a_n$ be a series w/ $a_n \geq 0$ for all $n \in \mathbb{N}$.

(i) If $\sum_{n=1}^{\infty} a_n$ converges and $|b_n| \leq a_n$, $\forall n \in \mathbb{N}$,

then $\sum_{n=1}^{\infty} b_n$ converges.

(ii) If $\sum_{n=1}^{\infty} a_n = +\infty$ and $b_n \geq a_n$, $\forall n \in \mathbb{N}$,

then $\sum_{n=1}^{\infty} b_n = +\infty$.

Pf. (i) $\sum_{n=1}^{\infty} a_n$ converges $\xRightarrow[\text{criterion}]{\text{Cauchy}}$ $\forall \varepsilon > 0$ $\exists N \in \mathbb{N}$ s.t.

$n \geq m > N \Rightarrow \sum_{k=m+1}^{n} a_k < \varepsilon$.

Since $\left| \sum_{k=m+1}^{n} b_k \right| \overset{\text{triangle ineq.}}{\leq} \sum_{k=m+1}^{n} |b_k| \overset{|b_k| \leq a_k}{\leq} \sum_{k=m+1}^{n} a_k$;

(no absolute value needed because $a_n \geq 0$)

$(|b_{m+1} + b_{m+2} + \dots + b_n| \leq |b_{m+1}| + |b_{m+2}| + \dots + |b_n|)$

we conclude $\left| \sum_{k=m+1}^{n} b_k \right| < \varepsilon$,

thus $\sum_{n=1}^{\infty} b_n$ converges, by the Cauchy criterion.

(ii) Let $s_n = \sum_{k=1}^{n} a_k$, $t_n = \sum_{k=1}^{n} b_k$ be

the sequences of partial sums.

$$\sum_{n=1}^{\infty} a_n = +\infty \iff \lim_{n\to\infty} s_n = +\infty$$

Since $b_k \geq a_k$ $\forall k$ $\Longrightarrow$ $t_n \geq s_n$ $\forall n$, by the above, we have $\lim_{n\to\infty} t_n = +\infty$; which, by definition, gives $\sum_{n=1}^{\infty} b_n = +\infty$. $\square$

<u>Corollary</u>. Absolute convergence $\Longrightarrow$ convergence

<u>Pf.</u> Apply (i) with $a_n = |b_n|$; to conclude that

$$\sum_{n=1}^{+\infty} a_n = \sum_{n=1}^{+\infty} |b_n| \text{ converges} \Longrightarrow \sum_{n=1}^{\infty} b_n \text{ converges.}$$ $\square$

$a_n = |b_n| \geq b_n$

<u>Root Test</u>.

Let $\sum_{n=1}^{+\infty} a_n$ be a series, and let $\alpha = \limsup_{n\to\infty} |a_n|^{\frac{1}{n}}$

(i) If $\alpha < 1$, then $\sum_{n=1}^{+\infty} a_n$ converges absolutely.

(ii) If $\alpha > 1$, then $\sum_{n=1}^{+\infty} a_n$ diverges

(iii) If $\alpha = 1$, then test is inconclusive ($\sum_{n=1}^{+\infty} a_n$ may diverge or converge).

Ratio Test.

Let $\sum_{n=1}^{+\infty} a_n$ be a series with $a_n \neq 0$, $\forall n \in \mathbb{N}$. Then:

(i) If $\limsup_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| < 1$, then $\sum_{n=1}^{+\infty} a_n$ converges absolutely

(ii) If $\liminf_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| > 1$, then $\sum_{n=1}^{+\infty} a_n$ diverges

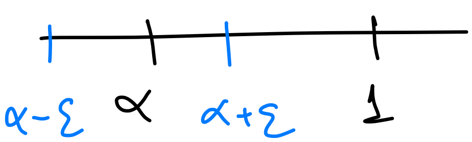
(iii) If $\liminf_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| \leq 1 \leq \limsup_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right|$, then the test is inconclusive ($\sum_{n=1}^{+\infty} a_n$ may diverge or converge.).

Proof (Root test) $\quad \alpha = \limsup_{n\to\infty} |a_n|^{\frac{1}{n}} = \frac{1}{R}$

$R =$ "radius of convergence"

(i) As $\alpha < 1$, there exists $\varepsilon > 0$ such that

$$\alpha + \varepsilon < 1.$$



As $\alpha = \lim_{N\to\infty} \sup\{|a_n|^{\frac{1}{n}} : n > N\}$, there exists $N \in \mathbb{N}$ such that

$$\alpha - \varepsilon < \sup\{|a_n|^{\frac{1}{n}} ; n > N\} < \alpha + \varepsilon < 1$$

Therefore $0 < |a_n|^{\frac{1}{n}} < \alpha + \varepsilon$ for all $n > N$.

Taking n^{th} power: $|a_n| < (\alpha+\varepsilon)^n < 1$ for all $n > N$.

Consider the Geometric Series $\sum_{n=N+1}^{+\infty} (\underbrace{\alpha+\varepsilon}_{r=\alpha+\varepsilon<1})^n$.

Since its ratio $r = \alpha+\varepsilon < 1$, thus geometric series converges. By Comparison Test:

$$\sum_{n=N+1}^{+\infty} |a_n| < \sum_{k=N+1}^{+\infty} (\alpha+\varepsilon)^n < +\infty$$

i.e. $\sum_{n=N+1}^{+\infty} |a_n| < +\infty$, and thus $\sum_{n=1}^{+\infty} |a_n| < \infty$,

because $\sum_{n=1}^{+\infty} |a_n| = \underbrace{\sum_{n=1}^{N} |a_n|}_{\text{finite number}} + \underbrace{\sum_{n=N+1}^{+\infty} |a_n|}_{\text{converges by the above}} < +\infty$.

(ii) If $\alpha = \limsup_{n\to\infty} |a_n|^{\frac{1}{n}} > 1$, there exists a subsequence $|a_{n_k}|$ such that $|a_{n_k}| \longrightarrow \alpha > 1$; in particular $|a_n| > 1$ for infinitely many n's.

so, if it exists, then $\lim_{n\to\infty} a_n \neq 0$. By the

n^{th} term test, it follows that $\sum_{n=1}^{+\infty} a_n$ diverges.

(iii) Test is inconclusive if $\alpha = 1$.

Examples: $\sum_{n=2}^{+\infty} \frac{1}{n} = +\infty$,

$$\alpha = \limsup_{n\to\infty} \left|\frac{1}{n}\right|^{\frac{1}{n}} = \lim_{n\to\infty} \frac{1}{\underbrace{n^{1/n}}_{\to 1}} = 1.$$

We will prove both of these claims next time

$$\sum_{n=1}^{+\infty} \frac{1}{n^2} < +\infty$$

$$\alpha = \limsup_{n\to\infty} \left|\frac{1}{n^2}\right|^{\frac{1}{n}} = \lim_{n\to\infty} \frac{1}{\underbrace{n^{2/n}}_{\to 1}} = 1.$$

b/c: $n^{2/n} = (\underbrace{n^{1/n}}_{\to 1})^2 \to 1$

$\square$

Proof (Ratio test).

Recall:

$$\liminf_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| \le \liminf_{n\to\infty} |a_n|^{1/n} \le \underbrace{\limsup_{n\to\infty} |a_n|^{1/n}}_{\alpha} \le \limsup_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right|.$$

(i) If $\limsup_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| < 1$, then $\alpha < 1$. So, the series $\sum_{n=1}^{+\infty} a_n$ converges absolutely by the Root Test.

(ii) If $\liminf_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| > 1$, then $\alpha > 1$, so the series $\sum_{n=1}^{+\infty} a_n$ diverges by the Root Test.

(iii) If $\liminf_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| \leq 1 \leq \limsup_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right|$, then test is inconclusive.

<u>Examples</u>: $\sum_{n=1}^{+\infty} \frac{1}{n} = +\infty$

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{\frac{1}{n+1}}{\frac{1}{n}} = \lim_{n\to\infty} \frac{n}{n+1} = 1.$$

$$\sum_{n=1}^{+\infty} \frac{1}{n^2} < +\infty$$

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{\frac{1}{(n+1)^2}}{\frac{1}{n^2}} = \lim_{n\to\infty} \frac{n^2}{n^2+2n+1} = 1.$$

$\square$

<u>Remark</u>: Note that Ratio Test <u>follows</u> from the Root test, i.e. Root test is the superior test.