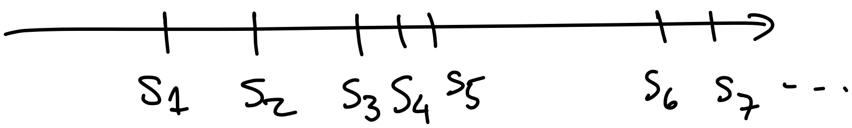
MAT320/640 Lecture 7 9/27/2021

Monotone Sequences

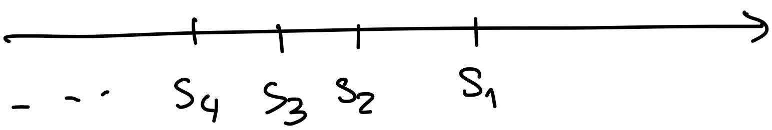
Def: A sequence $(s_n)_{n\in\mathbb{N}}$ is <u>monotone increasing</u> if $s_n \le s_{n+1}$ for all $n \in \mathbb{N}$,



(Note: If s_n is increasing, then: $s_n \le s_{n+1} \le s_{n+2} \le s_{n+3} \le \dots$ so we can equivalently write $n \le m \implies s_n \le s_m$.)

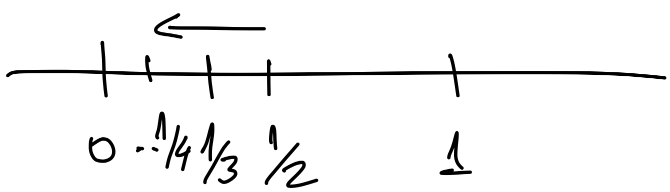
and it is <u>monotone decreasing</u> if $s_n \ge s_{n+1}$ for all $n\in\mathbb{N}$.

(... $n \le m \implies s_n \ge s_m$)



Collectively, these are called monotone / monotonic sequences.

Examples. $s_n = \dfrac{1}{n}$



✓ increasing

✗ decreasing

$$s_n = \frac{1}{n}, \quad s_{n+1} = \frac{1}{n+1}$$

$$s_{n+1} = \frac{1}{n+1} < \frac{1}{n} = s_n \quad \text{for all } n \in \mathbb{N}$$

This is a strictly decreasing sequence

$s_n = n^3$ increasing, $s_{n+1} = (n+1)^3 > n^3 = s_n$

$s_n = \left(1 + \frac{1}{n}\right)^n$

As increasing ... but why?

How do you compare:

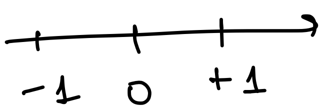
$$s_n = \left(1 + \frac{1}{n}\right)^n \quad \text{and} \quad s_{n+1} = \Big(\underbrace{1 + \frac{1}{n+1}}_{\text{smaller}}\Big)^{\overset{\text{larger}}{n+1}} \ ?$$

$$s_{n+1} = \left(1 + \frac{1}{n+1}\right)^n \cdot \Big(\underbrace{1 + \frac{1}{n+1}}_{>1}\Big)$$

Exercise: Prove that $s_{n+1} > s_n$.

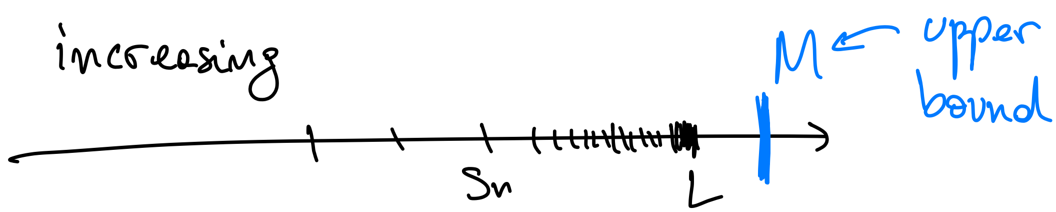
$$s_n = (-1)^n = \begin{cases} 1, & n \text{ is even} \\ -1, & n \text{ is odd} \end{cases}$$

not monotonic



Same for: $s_n = (-1)^n \cdot n$ or $s_n = \frac{(-2)^n}{n}$

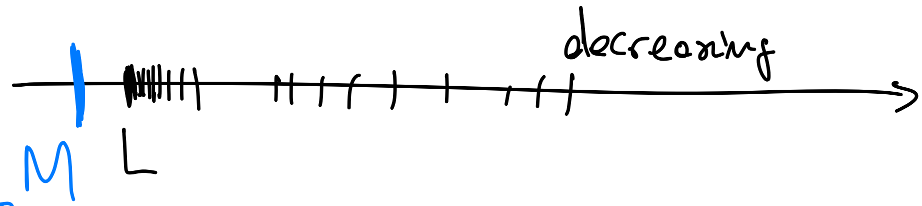
Theorem. A monotone increasing sequence which is bounded from above converges.



$s_n \le M$

$s_n \to L$

A monotone decreasing sequence which is bounded from below converges



lower bound

$$S_n \geq M, \qquad S_n \to L$$

<u>Proof</u>: S_n is monotone increasing, that is

if $n \leq m$, then $S_n \leq S_m$

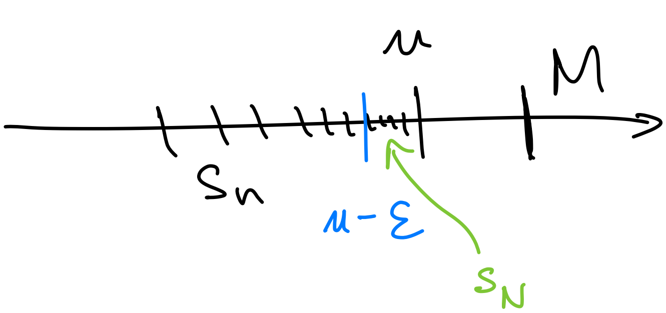
Know (hypothesis)

S_n is bounded from above, that is

$\exists M \in \mathbb{R}$ such that $S_n \leq M$ for all $n \in \mathbb{N}$.

Then the set $\{S_n : n \in \mathbb{N}\}$ is bounded from above, and thus has a supremum which is a real number:

$$u = \sup\{S_n : n \in \mathbb{N}\}$$



<u>Claim</u>: $S_n \to u$, that is

$\forall \varepsilon > 0, \exists N \in \mathbb{N}$

$n \geq N \Rightarrow |S_n - u| < \varepsilon$.

Given $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that $u - \varepsilon < s_N \leq u$. Since $(s_n)_{n\in\mathbb{N}}$ is increasing, if $n \geq N$, $s_n \geq s_N$. Thus, for all $n \geq N$,

$$u - \varepsilon < s_N \leq s_n \leq u = \sup\{s_n : n \in \mathbb{N}\} < u + \varepsilon$$

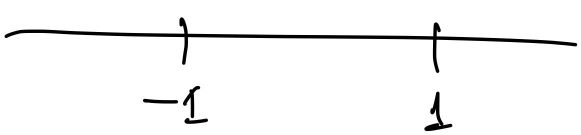
Thus $u - \varepsilon < s_n < u + \varepsilon$, i.e.

$$-\varepsilon < s_n - u < \varepsilon, \quad \text{i.e.} \quad |s_n - u| < \varepsilon.$$

$\square$

Q: If a sequence is only <u>bounded</u>, does it converge?

A: No; for example: $a_n = (-1)^n$ is <u>bounded</u>



$|a_n| \leq 1, \ \forall n \in \mathbb{N}$

and it does not converge.
(it is not monotonic)

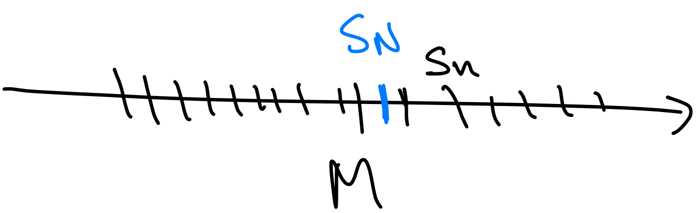
Q: If a sequence is only <u>monotonic increasing</u>, does it converge?

A: No, for example: $a_n = n$ is increasing and does not converge ($\lim a_n = +\infty$)
(it is <u>not</u> bounded from above)

Theorem. (i) If $(s_n)_{n\in\mathbb{N}}$ is unbounded from above, and monotone increasing, then $\lim_{n\to\infty} s_n = +\infty$.

(ii) If $(s_n)_{n\in\mathbb{N}}$ is unbounded from below, and monotonic decreasing, then $\lim_{n\to\infty} s_n = -\infty$.

Proof. (i) Given $M\in\mathbb{R}$, since the set $\{s_n : n\in\mathbb{N}\}$ is unbounded, there exists some $N\in\mathbb{N}$ such that $s_N \geq M$. Since the sequence is increasing, $s_n \geq s_N$ for all $n\geq N$.



$\exists N\in\mathbb{N}$ $n\geq N \Rightarrow s_n \geq M$

Thus, $\forall n\geq N$,

$$s_n \geq s_N \geq M;$$

which means that $\lim_{n\to\infty} s_n = +\infty$.

$\square$

Altogether: Monotone increasing sequences s_n either

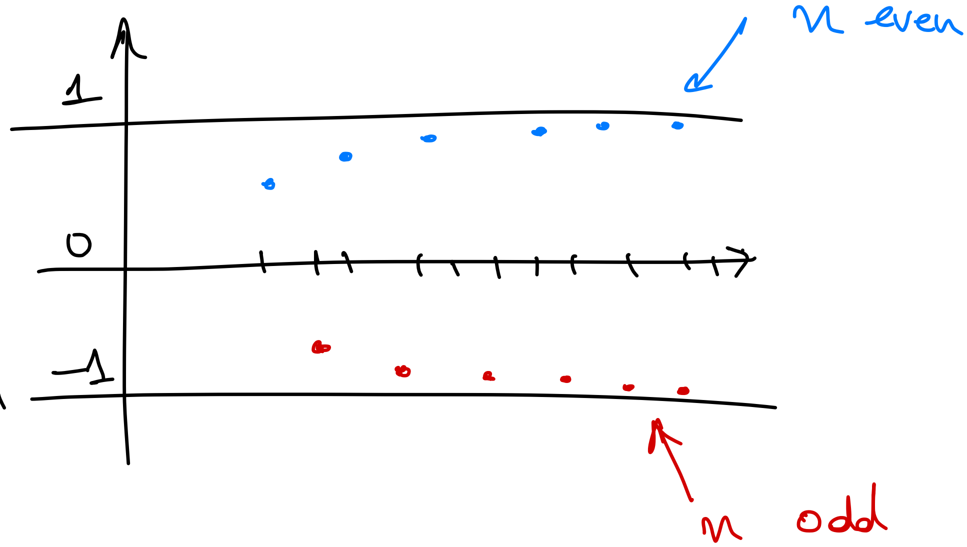
- converge (if bounded from above)

$$s_n \longrightarrow L$$

- diverge to $+\infty$ (if unbounded).

$$s_n \longrightarrow +\infty$$

$$s_n = \frac{(-1)^n}{1+\frac{1}{n}}$$



$$= \begin{cases} \dfrac{1}{1+\frac{1}{n}} & n \text{ even} \\[2ex] \dfrac{-1}{1+\frac{1}{n}} & n \text{ odd} \end{cases}$$

Does s_n converge? <u>No.</u>

$$\liminf_{n\to\infty} s_n = -1, \quad \limsup_{n\to\infty} s_n = +1$$

<u>Def</u>: Given a sequence $(s_n)_{n\in\mathbb{N}}$, consider the sets

$$\{s_n : n \geqslant N\}$$

infinitely many indices n satisfy this!

Then define:

$$\limsup_{n\to\infty} s_n := \lim_{N\to\infty} \sup \{s_n : n \geqslant N\}$$

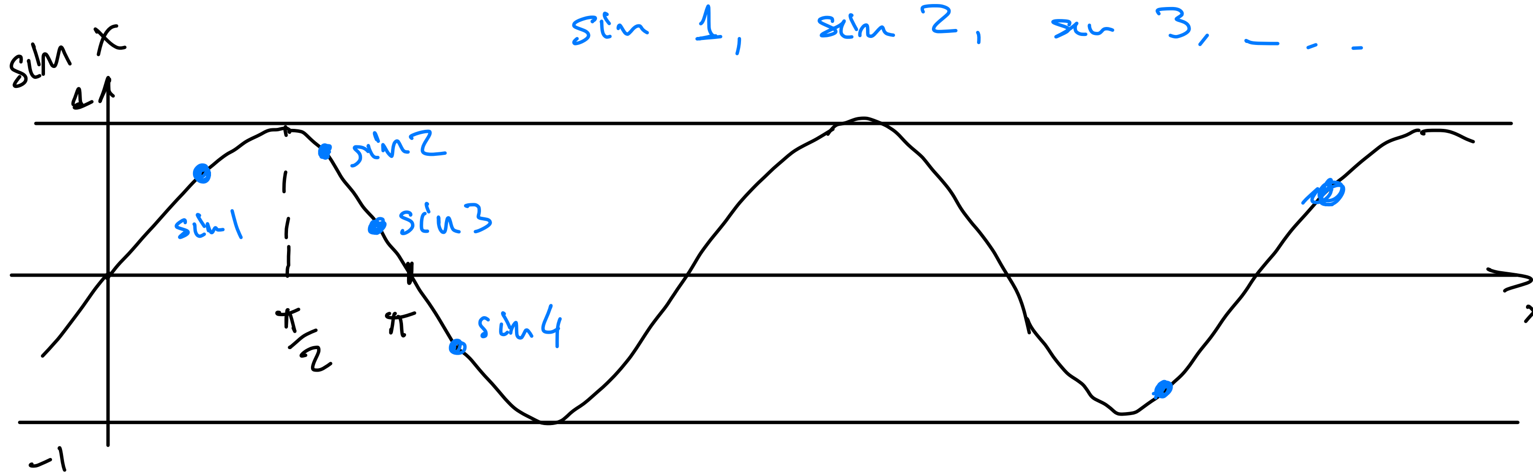
$$\liminf_{n\to\infty} s_n := \lim_{N\to\infty} \inf \{s_n : n \geqslant N\}.$$

"The $\limsup_{n\to\infty} s_n$ is the largest number which is approximated infinitely often along the sequence s_n".

<u>Ex</u>: $$\limsup_{n\to\infty} \sin(n) = 1$$

$$\liminf_{n\to\infty} \sin(n) = -1.$$

$\sin 1, \sin 2, \sin 3, \ldots$

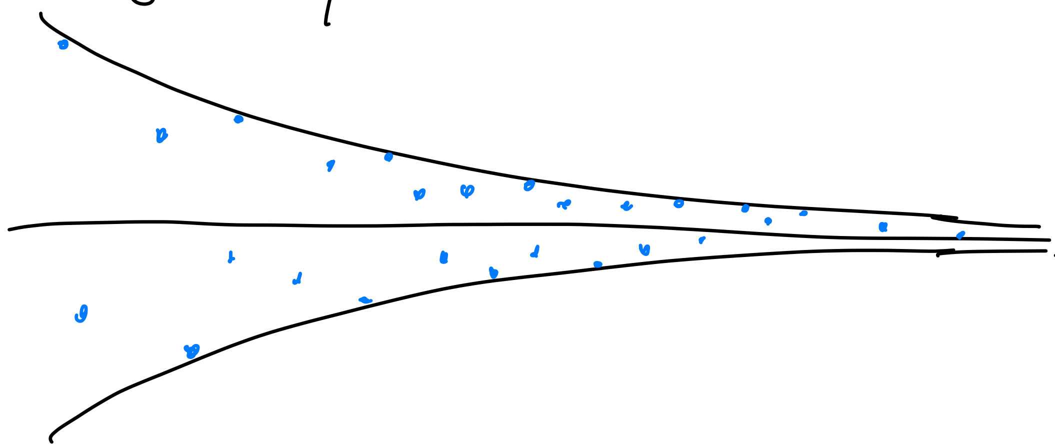


$$\lim_{n\to\infty} \sin(n) \quad \text{does not exist.}$$

<u>Theorem</u>. Given a sequence $(s_n)_{n\in\mathbb{N}}$, $\liminf_{n\to\infty} s_n \le \limsup_{n\to\infty} s_n$; with equality if and only if $(s_n)_{n\in\mathbb{N}}$ converges; in which

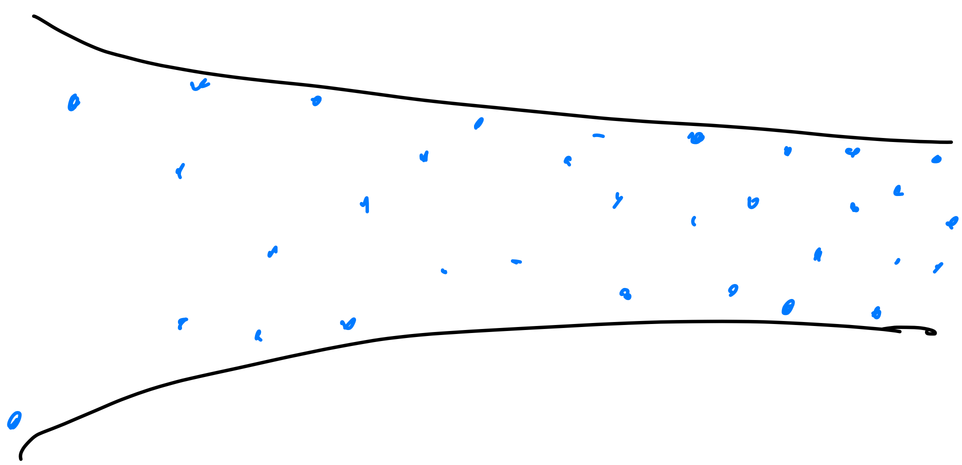
case $\lim_{n\to\infty} s_n = \liminf_{n\to\infty} s_n = \limsup_{n\to\infty} s_n.$

Converging sequence:

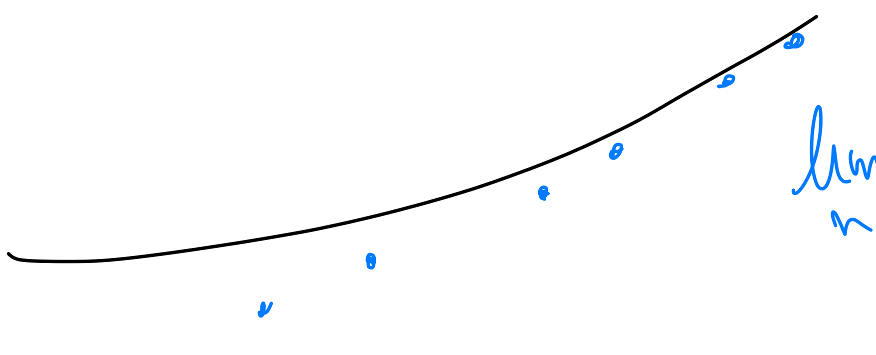


$$\limsup_{n\to\infty} s_n = \lim_{n\to\infty} s_n = \liminf_{n\to\infty} s_n$$

Sequence that does not converge



$$\limsup_{n\to\infty} s_n > \liminf_{n\to\infty} s_n$$



$$\limsup_{n\to\infty} s_n = +\infty$$

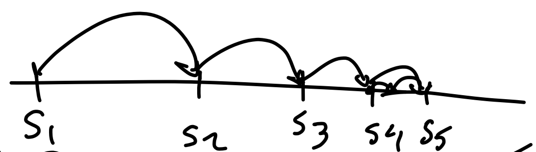
Proof: Read on Ross p. 61-62.

Cauchy sequence.

Def. A sequence $(s_n)_{n\in\mathbb{N}}$ is Cauchy if $\forall \varepsilon > 0 \; \exists N \in \mathbb{N}$ such that if $n, m \geq N$, then $\underbrace{|s_n - s_m| < \varepsilon}$.

distance between elements of the sequence gets arbitrarily small, provided we are "far enough" along the sequence.

"The length of each jump shrinks each time"



Proposition. Convergent sequences are Cauchy.

Pf: If $(s_n)_{n\in\mathbb{N}}$ converges, then $\exists L \in \mathbb{R}$ such that $\forall \tilde{\varepsilon} > 0 \; \exists \tilde{N} \in \mathbb{N}$ s.t. if $n \geq \tilde{N}$ then $|s_n - L| < \tilde{\varepsilon}$. Given $\varepsilon > 0$, applying the def. of convergence with $\tilde{\varepsilon} = \frac{\varepsilon}{2}$, we find that $\exists \tilde{N} \in \mathbb{N}$

$$|s_n - L| < \frac{\varepsilon}{2} \quad \text{for all } n \geq \tilde{N}.$$

So for $n, m \geq \tilde{N}$, we have:

$$|s_n - s_m| = |\underbrace{s_n - L}_{a} - (\underbrace{s_m - L}_{b})|$$

triangle ineq.

$|a-b| \leq |a|+|b|.$

$$\leq \underbrace{|s_n - L|} + \underbrace{|s_m - L|}$$

$$< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

This means $(s_n)_{n\in\mathbb{N}}$ is Cauchy. $\square$