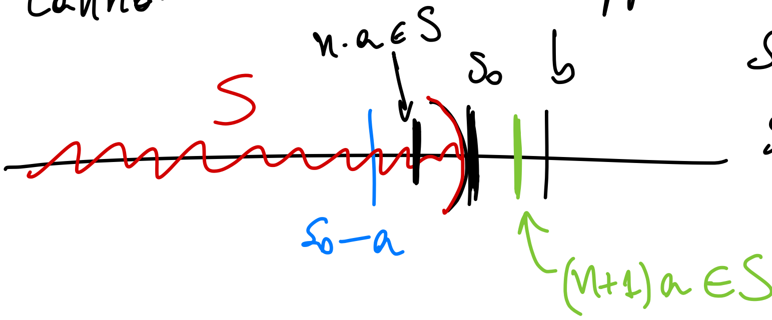
<u>Archimedean property</u>. If $a>0$ and $b>0$, then there exists $n\in\mathbb{N}$ such that $n\cdot a>b$.

"No matter how small $a>0$ is, and how large $b>0$ is, there always is some $n\in\mathbb{N}$ such that $n\cdot a>b$."

"No matter how small a spoon is, and how large a bathtub is, eventually you can take all the water out using the spoon."

<u>Proof</u>: (by Contradiction). Suppose that the Archimedean property fails: there exist $a>0$ and $b>0$ such that $n\cdot a\leq b$ for all $n\in\mathbb{N}$. This means that the set $S=\{n\cdot a : n\in\mathbb{N}\}$ is bounded from above by b. By the Completeness Axiom, there exists $s_0=\sup S$, the least upper bound of S. In particular, $s_0\leq b$. Moreover, $s_0-a<s_0$, so s_0-a cannot be an upper bound for S, i.e., there exists some $n\cdot a\in S$, with $n\in\mathbb{N}$ such that



$$s_0-a<\underbrace{n\cdot a}_{\in S}\leq s_0$$

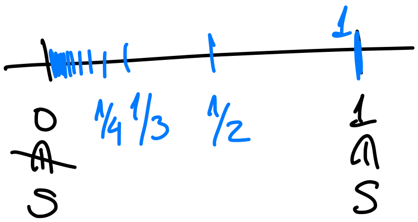
Therefore, $s_0<a+n\cdot a=(n+1)a\in S$. This contradicts the fact that s_0 is an upper bound for S. Thus the Archimedean property holds. $\square$

Examples of sup and inf using Archimedean prop.

1) $S = \left\{ \frac{1}{n} : n \in \mathbb{N} \right\}$ $= \left\{ 1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \dots \right\}$

Bounded from below? Yes Bounded from above? Yes

$\inf S = \underline{0}$, $\sup S = \underline{1}$.



A: The set S is bounded from below by 0, since $\frac{1}{n} > 0$ for all $n \in \mathbb{N}$.
The set S is bounded from above by 1, since $\frac{1}{n} \leq 1$ for all $n \in \mathbb{N}$.

The supremum of S is $\sup S = 1$, because $\frac{1}{n} \leq 1$ for all $n \in \mathbb{N}$ and $\frac{1}{1} = 1 \in S$. The infimum of S is $\inf S = 0$. Indeed, $0 < \frac{1}{n}$ for all $n \in \mathbb{N}$ (i.e., 0 is a lower bound). To prove 0 is the largest lower bound for S, we argue by contradiction: suppose it is not; i.e., suppose there exists $a > 0$ which is a lower bound for S, that is, $0 < a \leq \frac{1}{n}$ for all $n \in \mathbb{N}$. Take such $a > 0$ and $b = 1$, by the Archimedean property there exists $n_0 \in \mathbb{N}$ such that $n_0 a > b = 1$, which contradicts the above statement that $a \leq \frac{1}{n}$ for all $n \in \mathbb{N}$. This proves that $\inf S = 0$.

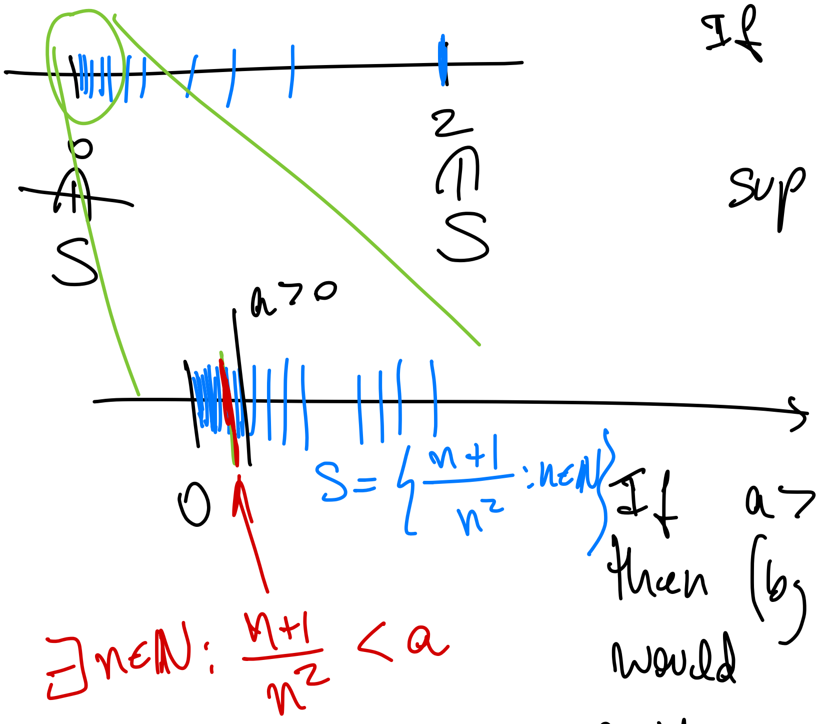
2) $S = \left\{ \frac{n+1}{n^2} : n \in \mathbb{N} \right\}$

Bounded from below? Yes Bounded from above? Yes

$\inf S = 0$, $\sup S = 1$

For all $n \in \mathbb{N}$, $0 < \frac{n+1}{n^2} = \frac{1}{n} + \frac{1}{n^2} \leq 1 + 1 = 2$

lower bound for S upper bound for S



If $n=1$, $\frac{n+1}{n^2} = \frac{1+1}{1^2} = 2$.

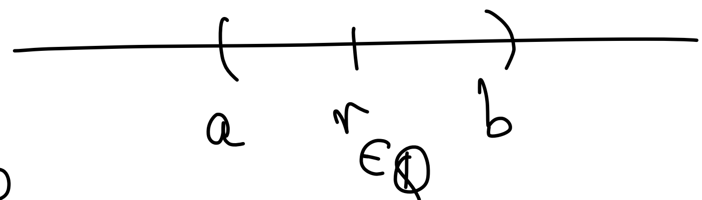
$\sup S = 2$

$\inf S = 0$, because:

If $a>0$ was a lower bound for S, then (by Archimedean property) there would exist $n \in \mathbb{N}$ such that $\underbrace{\frac{n+1}{n^2}}_{\in S} < a$, contradicting the assumption that $a>0$ is a lower bound for S.

Density of $\mathbb{Q}$ in $\mathbb{R}$

Theorem: For all $a<b$ in $\mathbb{R}$, there exists $r \in \mathbb{Q}$ such that $a<r<b$.



Proof: We must show that there exist $m, n \in \mathbb{Z}$, $n>0$, such that $r = \frac{m}{n} \in (a,b)$, that is,

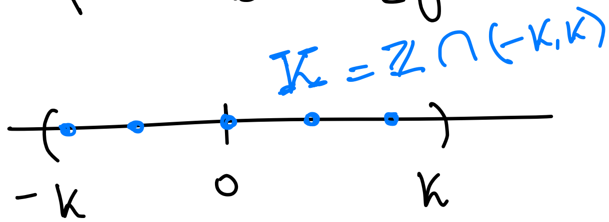
$$a < \frac{m}{n} < b \iff n \cdot a < m < n \cdot b$$

Since $b-a>0$, by the Archimedean property (applied to $b-a$ and 1), there exists $n \in \mathbb{N}$ such that $n \cdot (b-a) > 1$; that is, $n \cdot b - n \cdot a > 1$. $(*)$

By the Archimedean property, there exists $k \in \mathbb{N}$ such that $k > |an|$ and $k > |bn|$; that is,

$$-k < an < bn < k$$

Define $\mathbb{K} := \{ j \in \mathbb{Z} : -k \leq j \leq k \}$ and $\{ j \in \mathbb{K} : a \cdot n < j \}$



Both of these are finite sets, so they admit minima and maxima.

Let $m = \min \{ j \in \mathbb{K} : a \cdot n < j \}$. Then $-k < a \cdot n < m$ (**) and $-k, m \in \mathbb{Z}$, so $(m-1) \in \mathbb{K}$, but $(m-1) \notin \{ j \in \mathbb{K} : a \cdot n < j \}$. Thus, $a \cdot n \geq m - 1$, i.e. $m - 1 \leq a \cdot n$. Put the inequalities together:

(last ineq. in previous page)

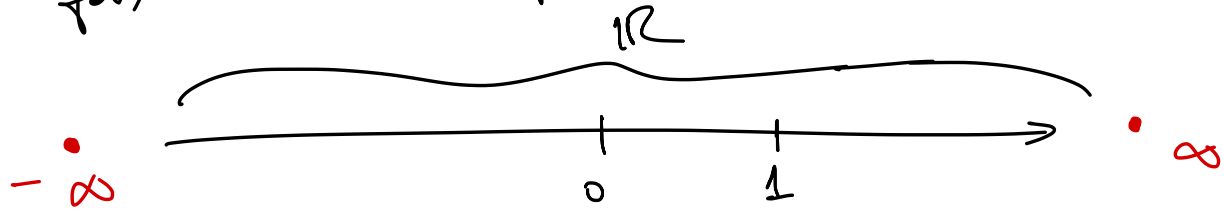
$$a \cdot n \overset{(**)}{<} m \leq a \cdot n + 1 \overset{(*)}{<} b \cdot n$$

which is what we needed, to conclude $r = \frac{m}{n} \in (a, b)$.

$\square$

Infinities and the "extended" real line

So far, we have defined (via Axioms) the real line $\mathbb{R}$



which is an ordered field and complete.

(algebraic structure)

Completeness axiom ("no gaps")

It is often useful to consider $+\infty$, $-\infty$, and the "extended" real line $\mathbb{R} \cup \{-\infty, +\infty\}$. Beware that the <u>field structure</u> (i.e., addition and multiplication) <u>do not</u> extend to $\{\pm\infty\}$, even though the <u>order</u> does:

"For all $a \in \mathbb{R}$, $-\infty < a < +\infty$."

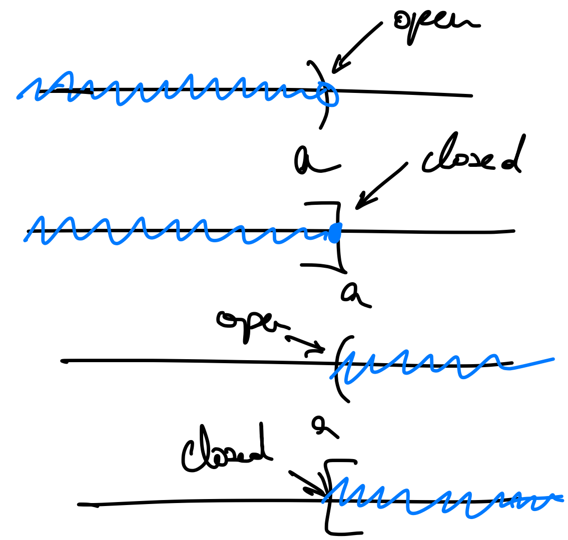
Accordingly, we may work with <u>unbounded</u> intervals:

$$(-\infty, a) = \{x \in \mathbb{R} : x < a\}$$

$$(-\infty, a] = \{x \in \mathbb{R} : x \leq a\}$$

$$(a, +\infty) = \{x \in \mathbb{R} : x > a\}$$

$$[a, +\infty) = \{x \in \mathbb{R} : x \geq a\}$$



$$(-\infty, +\infty) = \mathbb{R}$$

<u>Note</u>: intervals are always <u>open</u> at the infinite endpoint.

<u>Example</u>: For all $a, b \in \mathbb{R}$,

bounded intervals

$$\inf (a,b) = \inf (a,b] = \inf [a,b] = \inf [a,b) = a$$

$$\sup (a,b) = \sup (a,b] = \sup [a,b] = \sup [a,b) = b$$

unbounded intervals

$$\begin{cases} \inf\,(-\infty, a) = \inf\,(-\infty, a] = -\infty \\ \sup\,(-\infty, a) = \sup\,(-\infty, a] = a \\ \inf\,(a, +\infty) = \inf\,[a, +\infty) = a \\ \sup\,(a, +\infty) = \sup\,[a, +\infty) = +\infty. \end{cases}$$

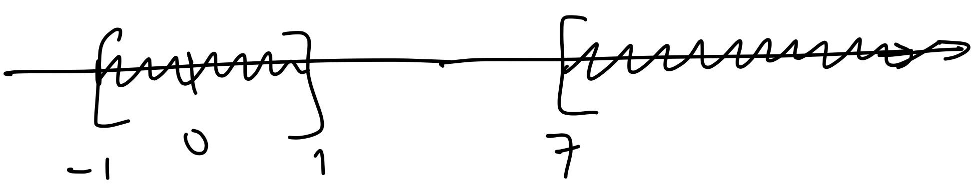
Just like the above, we extend the definition of inf and sup to subsets of $\mathbb{R} \cup \{\pm\infty\}$ by setting

$$\inf S := -\infty \quad \text{if } S \text{ is unbounded from below}$$

$$\sup S := +\infty \quad \text{if } S \text{ is unbounded from above.}$$

<u>Example:</u> $S = \{x \in \mathbb{R} : 1 - x^2 \geq 0 \text{ or } x \geq 7\}$

i.e.,



$S = [-1, 1] \cup [7, +\infty)$

$\sup S = +\infty$, since S is unbounded from above

$(\inf S = -1)$