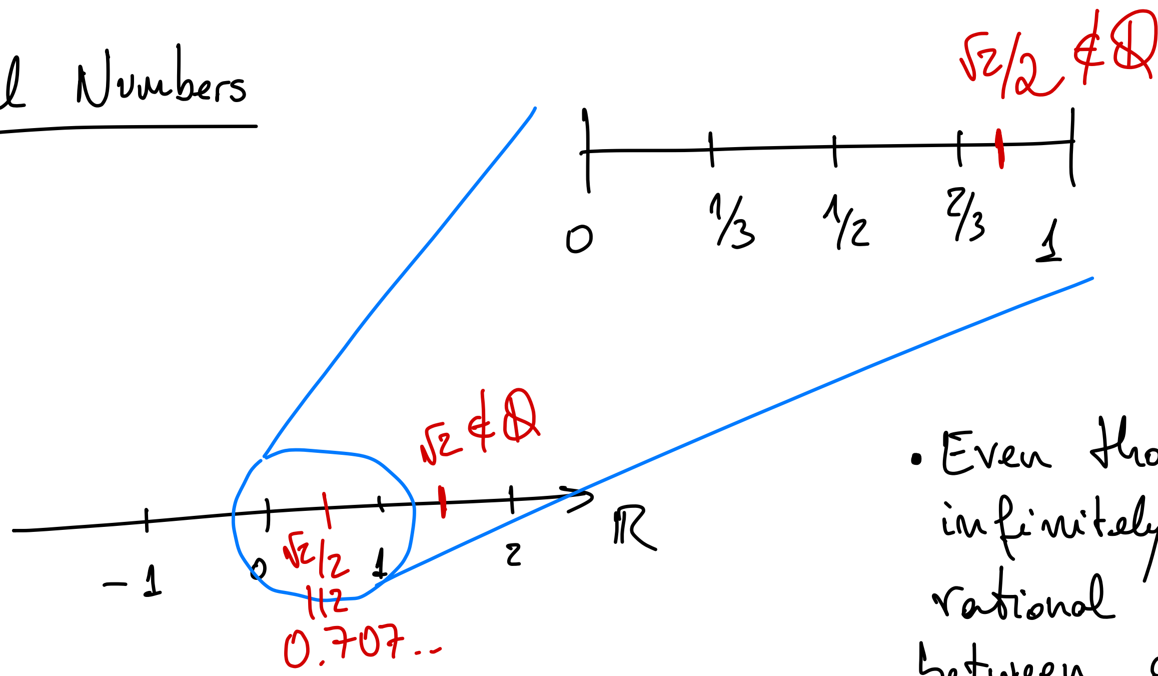
MAT320/640 Lecture 3 9/1/2021

Real Numbers



- Even though there are infinitely many rational numbers ($\mathbb{Q}$) between any two numbers, there are also infinitely many numbers "missing", i.e., numbers that are not in $\mathbb{Q}$.

$\mathbb{Q}$ ✓ $\rightsquigarrow$ $\mathbb{R}$? ← Defining this set rigorously is the goal for today.

Definition: A *field* is a set F with two binary operations, namely an addition $+ : F \times F \to F$, and a product $\cdot : F \times F \to F$, satisfying the following:

(A0) $x, y \in F, \quad x+y \in F$

(A1) $x+y = y+x$ ← commutative

(A2) $(x+y)+z = x+(y+z)$ ← associative

(A3) $\exists\, 0 \in F$ such that $0 + x = x, \quad \forall x \in F$

$\forall$ "for all"

$\exists$ "there exists"

(A4) $\forall x \in F, \ \exists -x \in F$ such that $x + (-x) = 0$.

(M0) $x, y \in F \qquad x \cdot y \in F$

(M1) $x \cdot y = y \cdot x \leftarrow$ commutative

(M2) $(x \cdot y) \cdot z = x \cdot (y \cdot z) \leftarrow$ associative

(M3) $\exists\, 1 \in F$ such that $1 \cdot x = x, \quad \forall x \in F$

(M4) $\forall x \in F, \quad x \neq 0, \quad \exists\, \frac{1}{x} \in F$ such that $x \cdot \frac{1}{x} = 1$

(DL) $x \cdot (y + z) = x \cdot y + x \cdot z \leftarrow$ distributive law.

<u>Examples</u>: The <u>rational numbers</u> $\mathbb{Q}$ form a field.

The <u>integer numbers</u> $\mathbb{Z}$ do not satisfy (M4) and hence do not form a field.

<u>Definition</u>: A field F is called an <u>ordered field</u> if it has an order structure $\leq$ satisfying:

(O1) $\forall x, y \in F$, either $x \leq y$ or $y \leq x$

(O2) If $x \leq y$ and $y \leq x$, then $x = y$.

(O3) If $x \leq y$ and $y \leq z$, then $x \leq z$

(O4) If $x \leq y$, then $x + z \leq y + z, \quad \forall z \in F$

(O5) If $x \leq y$, and $0 \leq z$, then $xz \leq yz$.

<u>Examples</u>: $\mathbb{Q}$ is an <u>ordered field</u>.

($\mathbb{R}$ will also be an ordered field)

Theorem: If F is a field, then

(i) $x + z = y + z$ implies $x = y$

(ii) $x \cdot 0 = 0$ for all $x \in F$

(iii) $(-x) \cdot y = -(x \cdot y)$ for all $x, y \in F$

(iv) $(-x) \cdot (-y) = x \cdot y$ for all $x, y \in F$

(v) $x \cdot z = y \cdot z$ and $z \neq 0$, then $x = y$

If F is an ordered field, then:

$\overline{(i)}$ $x \leq y$ implies $-y \leq -x$

$\overline{(ii)}$ if $x \leq y$ and $z \leq 0$ then $yz \leq xz$

$\overline{(iii)}$ if $0 \leq x$ and $0 \leq y$, then $0 \leq x \cdot y$

$\overline{(iv)}$ for all $x \in F$, $0 \leq x^2$

$\overline{(v)}$ $0 \leq 1$ and $0 \neq 1$, i.e., $0 < 1$.

$\overline{(vi)}$ if $0 < x$, then $0 < \frac{1}{x}$

$$\boxed{\begin{array}{c} a \leq b \text{ and } a \neq b \\ \Updownarrow \\ a < b \end{array}}$$

$\overline{(vii)}$ if $0 < x < y$, then $0 < \frac{1}{y} < \frac{1}{x}$.

Lastly, there are no divisors of zero:

if $x \cdot y = 0$, then $x = 0$ or $y = 0$.

Proof (of just some of the statements)

$$x \cdot y = 0 \implies x = 0 \text{ or } y = 0$$

Pf: Suppose $x \cdot y = 0$. If $x = 0$, then we are done. Thus, assume $x \neq 0$. Then by (M4), there exists $\frac{1}{x} \in F$ such that $\frac{1}{x} \cdot x = 1$. Multiply both sides of $x \cdot y = 0$ by $\frac{1}{x}$ to find

$$y \overset{(M3)}{=} \underbrace{\left(\frac{1}{x} \cdot x\right)}_{\substack{\shortparallel \\ 1}} \cdot y \overset{(M2)}{=} \frac{1}{x} \cdot (x \cdot y) \overset{\downarrow}{=} \frac{1}{x} \cdot 0 \overset{(ii)}{=} 0$$

thus $y = 0$. $\square$

(A0) $x, y \in F, \quad x + y \in F$ **Field**

(A1) $x + y = y + x$

(A2) $(x+y) + z = x + (y+z)$

(A3) $\exists 0 \in F$ such that $0 + x = x, \quad \forall x \in F$

(A4) $\forall x \in F, \ \exists -x \in F$ such that $x + (-x) = 0$.

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(DL) $x \cdot (y+z) = x \cdot y + x \cdot z$

(O1) $\forall x, y \in F$, either $x \leq y$ or $y \leq x$

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(O4) If $x \leq y$, then $x + z \leq y + z, \quad \forall z \in F$

(O5) If $x \leq y$, and $0 \leq z$, then $xz \leq yz$.

ordered field

(i) $x + z = y + z \implies x = y$

By (A4), there exists $(-z) \in F$ such that $z + (-z) = 0$. Adding $(-z)$ to both sides of $x + z = y + z$, we find

$(x+z) + (-z) = (y+z) + (-z)$. By (A1) and (A2),

$$(x+z) + (-z) = x + (z + (-z)) = x + 0 \overset{(A3)}{=} x$$

$$(y+z) + (-z) = y + (z + (-z)) = y + 0 \overset{(A3)}{=} y$$

Altogether, we conclude that $x = y$. $\square$

(v) $0 < 1$, i.e., $0 \leq 1$ and $0 \neq 1$

<u>Proof:</u> To prove $0 \leq 1$, we use (iv) and (M3). By (M3), $1 \cdot 1 = 1$ and by (iv), $0 \leq 1^2 = 1 \cdot 1 = 1$. Thus $0 \leq 1$. To prove $0 \neq 1$, we use a proof by contradiction. Suppose that $0 = 1$, then (M3) we have if $x \neq 0$

$$x \overset{(M3)}{=} 1 \cdot x \overset{0=1}{=} 0 \cdot x \overset{(ic)}{=} 0$$

Thus $x = 0$, a contradiction with $x \neq 0$. Therefore $0 \neq 1$. Altogether we find $0 < 1$. □

Field

(A0) $x, y \in F$, $x + y \in F$

(A1) $x + y = y + x$

(A2) $(x+y)+z = x + (y+z)$

(A3) $\exists 0 \in F$ such that $0 + x = x$, $\forall x \in F$

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(M0) $x, y \in F$ $x \cdot y \in F$

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(M3) $\exists 1 \in F$ such that $1 \cdot x = x$, $\forall x \in F$

(M4) $\forall x \in F$, $x \neq 0$, $\exists \frac{1}{x} \in F$ such that $x \cdot \frac{1}{x} = 1$

(DL) $x \cdot (y+z) = x \cdot y + x \cdot z$

(O1) $\forall x, y \in F$, either $x \leq y$ or $y \leq x$

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ordered field

<u>Absolute value</u>

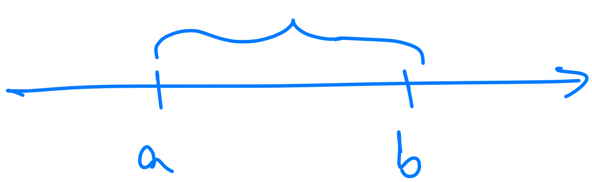
<u>Definition:</u> The <u>absolute value</u> of a is

$$|a| := \begin{cases} a & \text{if } a \geq 0 \\ -a & \text{if } a < 0 \end{cases}$$

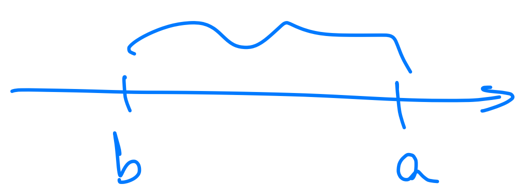
$a \leq 0$ and $a \neq 0$

<u>Remark:</u> The absolute value of real numbers is used to define a notion of distance between real numbers:

$$\mathrm{dist}(a,b) = |b - a|$$

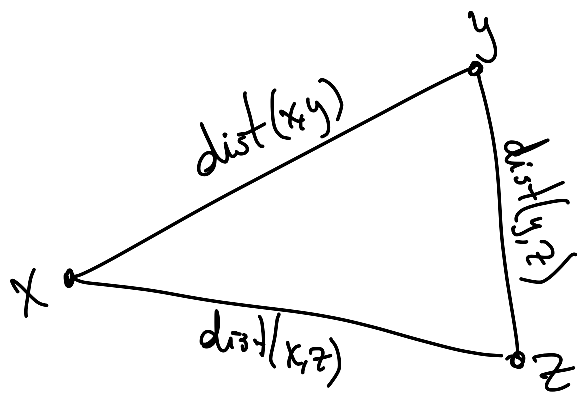


$$\mathrm{dist}(a,b) = |b - a| = |a - b|$$



Theorem: Thm 3.5 in Ross

- $|a| \geq 0$
- $|ab| = |a| \cdot |b|$
- Triangle inequality: $|a+b| \leq |a| + |b|$



$$\underbrace{\text{dist}(x,z)}_{\substack{\| \\ |x-z| \\ |a+b|}} \leq \underbrace{\text{dist}(x,y)}_{\substack{\| \\ |x-y| \\ |a|}} + \underbrace{\text{dist}(y,z)}_{\substack{\| \\ |y-z| \\ |b|}}$$

Use $a = x-y$
$b = y-z$

$a+b = (x-y)+(y-z)$
$= x-z$

Corollary: "Generalized" Triangle inequality: for all $n \in \mathbb{N}$

$$|a_1 + a_2 + \dots + a_n| \leq |a_1| + |a_2| + \dots + |a_n|$$

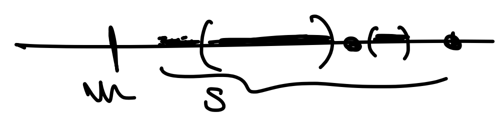
Pf: Exercise (using induction).

The Completeness Axiom ← ensuring there are no "gaps" in $\mathbb{R}$, like in $\mathbb{Q}$.

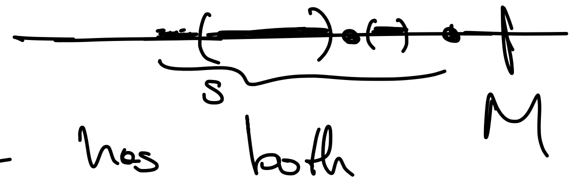
For now, assume $\mathbb{R}$ was already defined ...

Definition (Upper/Lower bound). Let $S \subset \mathbb{R}$ be a non empty subset.

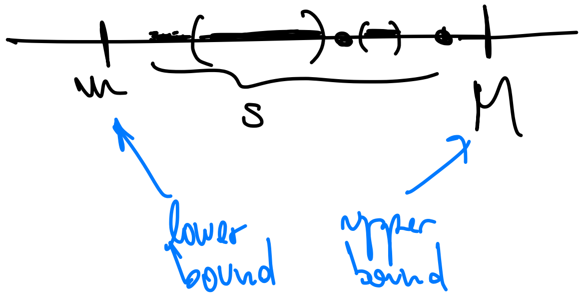
- A lower bound for S is a number $m \in \mathbb{R}$ such that $m \leq s$ for all $s \in S$.



- An <u>upper bound</u> for S is a number $M \in \mathbb{R}$ such that $s \leq M$ for all $s \in S$



- The set S is <u>bounded</u> if it has both a lower and an upper bound.



<u>Example:</u> If S has a minimum, $\min S$, i.e., an element $(\min S) \in S$ such that $\min S \leq s$, $\forall s \in S$, then $\min S$ is a lower bound for S. Similarly, if S has a maximum, $\max S$, then $\max S$ is an upper bound for S.

<u>However:</u> many sets S <u>do</u> <u>not</u> have minima or maxima, but <u>do</u> have lower or upper bounds.

For example $S = (0,1) = \{x \in \mathbb{R} : 0 < x < 1\}$

$\min S$ does not exist; nor does $\max S$

(because $0 \notin S$ and $1 \notin S$)

But $m = 0$ is a lower bound for S

and $M = 1$ is an upper bound for S.

<u>Definition</u> (Least upper bound / Largest lower bound)

Least upper bound: sup "supremum"

Largest lower bound: inf "infimum"

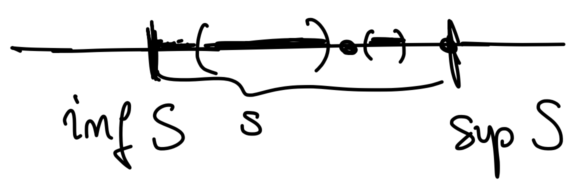
Given a nonempty subset $S \subset \mathbb{R}$

- if S is bounded from above, i.e., if S has an upper bound, then define the supremum of S to be

$$\sup S := \text{the least upper bound of } S$$

- if S is bounded from below, i.e., if S has a lower bound, then define the infimum of S to be

$$\inf S := \text{the largest lower bound of } S.$$



<u>Example</u>: $S = (0,1) = \{x \in \mathbb{R} : 0 < x < 1\}$

$$\inf S = 0$$

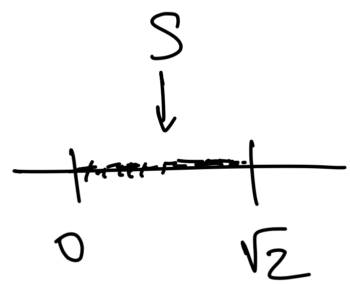
$$\sup S = 1$$

<u>Completeness Axiom</u>. Every nonempty subset S of $\mathbb{R}$ that is bounded from above has a supremum $\sup S \in \mathbb{R}$.

<u>Q</u>: What about $\sqrt{2} \notin \mathbb{Q}$ ...

<u>A</u>: Consider the following subset

$$S = \{ x \in \mathbb{Q} : x > 0, x^2 \le 2 \}$$



by Completeness Axiom

Since <u>S is bounded from above</u>, ← for example $M = 5$ is an upper bound.

$\sup S$ exists and is a real number,

That number is $\sqrt{2}$, i.e. $\underline{\sup S = \sqrt{2} \in \mathbb{R}}$.

<u>Theorem</u>: There exists a unique <u>ordered field</u> containing $\mathbb{Q}$ that satisfies the completeness axiom, called $\mathbb{R}$.

We will not prove this, and we will instead use it as the <u>Definition</u> of real numbers $\mathbb{R}$.

<u>Note</u>: It also follows from the Completeness Axiom that every non empty subset $S \subset \mathbb{R}$ which is bounded from below has an infimum $\inf S \in \mathbb{R}$.

(Cor. 4.5 in Ross)