Final Review

- About the Final Exam
- Review problems

1) Find all the rational roots of the polynomial
$p(x) = x^8 - 4x^5 + 13x^3 - 7x + 1$.

2) Prove that the sequence defined recursively by
$S_1 = 1$ and $S_{n+1} = \left(1 - \frac{1}{4n^2}\right) S_n$, $\forall n \in \mathbb{N}$, is Cauchy.

3) Find liminf and limsup of the sequences below:

a) $\frac{1}{2}, 0, 0, \frac{1}{4}, 0, 0, \frac{1}{8}, 0, 0, \frac{1}{16}, 0, 0, \ldots$

b) $b_n = \begin{cases} \frac{3n+2}{2n} & \text{if } n \text{ odd} \\ \frac{n-2}{2n} & \text{if } n \text{ even} \end{cases}$

c) $(-7)^n$

Do they converge?

4) For what values of $x \in \mathbb{R}$ does $\sum_{n=0}^{+\infty} \frac{x^n}{7^n}$ converge absolutely?

5) Let $f_n : [0, +\infty) \to \mathbb{R}$ be given by $f_n(x) = \frac{x^n}{n + x^n}$.

a) Does f_n converge pointwise to some $f(x)$? Where?

b) ——— " ——— uniformly ——— " ——— ? Where?

c) Compute $\lim_{n \to \infty} \int_0^1 f_n(x)\, dx$.

6) Same exercise as 5) but with $f_n(x) = \frac{1}{1 + x^n}$

Solutions

1) Find all the rational roots of the polynomial

$p(x) = 0$

$$p(x) = x^8 - 4x^5 + 13x^3 - 7x + 1.$$

The polynomial $p(x)$ is monic, has integer coefficients, and constant coefficient is 1. By the Rational Zeros Theorem, all the rational roots are among the divisors of 1, i.e. ± 1. Let us check:

$$p(1) = 1 - 4 + 13 - 7 + 1 = 15 - 11 = 4 \neq 0$$

$$p(-1) = 1 + 4 - 13 + 7 + 1 = 0$$

Therefore $x = -1$ is the only rational root of $p(x)$.

Note: If $p(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$ is not monic,

$a_n \neq 0$

consider $\frac{1}{a_n} \cdot p(x) = x^n + \frac{a_{n-1}}{a_n} x^{n-1} + \cdots + \frac{a_1}{a_n} x + \frac{a_0}{a_n}$ $\in \mathbb{Q}$

monic!

Rational Zeros Thm. If the polynomial

$$p(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$$

has integer coefficients $a_i \in \mathbb{Z}$ and $r = \frac{c}{d} \in \mathbb{Q}$ is a root of p, i.e., $p(r) = 0$, then

c divides a_0 and d divides a_n.

---

2) Prove that the sequence defined recursively by $s_1 = 1$ and $s_{n+1} = \left(1 - \frac{1}{4n^2}\right) s_n$, $\forall n \in \mathbb{N}$, is Cauchy.

NOTE: It is harder to prove this by directly verifying the definition of Cauchy: $\forall \varepsilon > 0$ $\exists N \in \mathbb{N}$ s.t. $n, m \geq N \implies |s_n - s_m| < \varepsilon$.

$$s_n = \left(1 - \frac{1}{4(n-1)^2}\right)\left(1 - \frac{1}{4(n-2)^2}\right) \cdots \left(1 - \frac{1}{4 \cdot 1^2}\right) \cancel{s_1}^{\,1}$$

$s_m = \cdots$

Note s_n is a monotonic decreasing sequence:

$$0 < \left(1 - \frac{1}{4n^2}\right) < 1 \Rightarrow \begin{cases} s_{n+1} = \underbrace{\left(1 - \frac{1}{4n^2}\right)}_{<1} s_n < s_n, \ \forall n \\ s_n > 0, \ \forall n \end{cases}$$

$\forall n \in \mathbb{N}$

So $s_{n+1} < s_n$ for all $n \in \mathbb{N}$. Furthermore, $s_n > 0$ is bounded from below. Therefore, by a Theorem from Lecture 7 (monotonic + bounded $\Rightarrow$ convergent), s_n converges. Since a sequence of real numbers converges if and only if it is Cauchy, it follows that s_n is Cauchy. $\square$

Remark: $\lim_{n\to\infty} s_n = \frac{2}{\pi} \cong 0.6366\ldots < 1$

3) Find liminf and limsup of the sequences below:

a) $\frac{1}{2}, 0, 0, \frac{1}{4}, 0, 0, \frac{1}{8}, 0, 0, \frac{1}{16}, 0, 0, \ldots$

b) $b_n = \begin{cases} \frac{3n+2}{2n} & \text{if } n \text{ odd} \\ \frac{n-2}{2n} & \text{if } n \text{ even} \end{cases}$

c) $(-7)^n$

Do they converge?

$\liminf_{n\to\infty} a_n :=$ smallest subsequential limit of $(a_n)_n$

$\limsup_{n\to\infty} a_n :=$ largest subsequential limit of $(a_n)_n$

$a_1,\ a_2,\ a_3,\ a_4,\ a_5,\ a_6,\ a_7,\ a_8,\ a_9,\ a_{10},\ a_{11},\ a_{12}$

a) $\frac{1}{2},\ \underline{0},\ \underline{0},\ \frac{1}{4},\ \underline{0},\ \underline{0},\ \frac{1}{8},\ \underline{0},\ \underline{0},\ \frac{1}{16},\ \underline{0},\ \underline{0},\ \cdots$

$n \equiv m \mod d$

n leaves remainder m when divided by d.

$$a_n = \begin{cases} \frac{1}{2^k} & \text{if } n \equiv 1 \mod 3 \text{ i.e. } n = 3k+1 \text{ for some } k = 0,1,2,\dots \\ 0 & \text{if } n \equiv 0,2 \mod 3 \text{ i.e. } n = 3k \text{ or } n = 3k+2 \text{ for some } k = 0,1,2,\dots \end{cases}$$

$\liminf_{n\to\infty} a_n = 0$ b/c $\underbrace{a_n \geq 0 \text{ for all } n \in \mathbb{N}}_{\Downarrow\ \liminf_{n\to\infty} a_n \geq 0}$

and $\underbrace{a_{3k} = 0 \text{ for all } k \in \mathbb{N}}_{\Downarrow\ \liminf_{n\to\infty} a_n \leq 0}$.

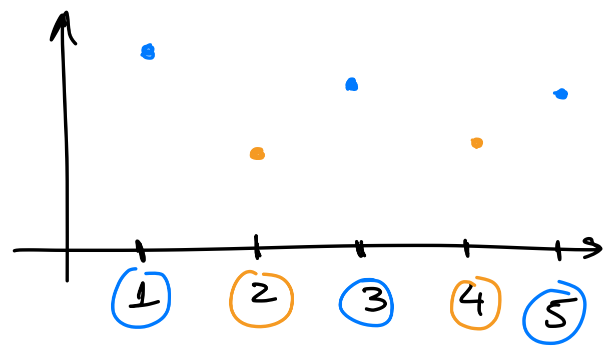
$\limsup_{n\to\infty} a_n = 0$ b/c a_n is either 0 or of the form $\frac{1}{2^k}$ and $\lim_{k\to\infty} \frac{1}{2^k} = 0$.

Therefore, the sequence a_n converges to 0.

Recall. a_n converges $\iff \liminf_{n\to\infty} a_n = \limsup_{n\to\infty} a_n$

In that case, $\lim_{n\to\infty} a_n = \liminf_{n\to\infty} a_n = \limsup_{n\to\infty} a_n$

b) $$b_n = \begin{cases} \frac{3n+2}{2n} & \text{if } n \text{ odd} \\ \frac{n-2}{2n} & \text{if } n \text{ even} \end{cases}$$



$$\liminf_{n\to\infty} b_n = \frac{1}{2}$$

$$\limsup_{n\to\infty} b_n = \frac{3}{2}$$

$\Longrightarrow$ b_n does not converge.

The subsequence of odd terms $b_{2k+1} = \frac{3(2k+1)+2}{2(2k+1)}$ converges to $3/2$ as $k\to\infty$, so

odd

$$\limsup_{n\to\infty} b_n \geq \frac{3}{2}.$$

The only other subsequences of b_n would include terms of the form b_{2k}, which are smaller,

even

hence $\limsup_{n\to\infty} b_n \leq \lim_{k\to\infty} b_{2k+1} = \frac{3}{2}.$

c) $c_n = (-7)^n$ $\quad \liminf_{n\to\infty} c_n = -\infty, \quad \limsup_{n\to\infty} c_n = +\infty.$

$$c_n = (-1)^n \cdot 7^n \implies |c_n| = 7^n \nearrow +\infty \quad \text{as } n \to \infty.$$

For liminf (resp. limsup) use a subsequence where n is odd (resp. even).

c_n does not converge.

4) For what values of $x \in \mathbb{R}$ does $\displaystyle\sum_{n=0}^{+\infty} \frac{x^n}{7^n}$ <u>converge</u> <u>absolutely</u>?

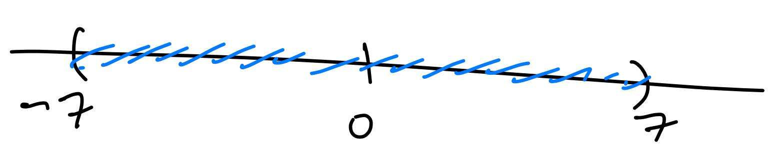
<u>Recall</u>: Radius of convergence: $\quad R = \dfrac{1}{\beta}$ where $\beta = \limsup_{n\to\infty} |a_n|^{1/n}$

$$|x - x_0| < R \implies \text{Series } \sum_{n=0}^{+\infty} a_n (x-x_0)^n \text{ converges absolutely.}$$

In the series above:

$$a_n = \frac{1}{7^n} \qquad \beta = \limsup_{n\to\infty} |a_n|^{1/n} = \limsup_{n\to\infty} \frac{1}{7} = \frac{1}{7}.$$

Radius of convergence is $R = \dfrac{1}{\beta} = 7$; so the series above converges absolutely if $|x| < 7$.



Endpoints: $x = \pm 7$, i.e., $|x| = 7$ then

$$\sum_{n=0}^{+\infty} \frac{7^n}{7^n} = \sum_{n=0}^{+\infty} 1 = +\infty$$ diverges.

So series converges absolutely if and only if $|x| < 7$.

Remark: $$\sum_{n=0}^{+\infty} \frac{x^n}{7^n} = \sum_{n=0}^{+\infty} \left(\frac{x}{7}\right)^n \overset{\text{Geometric Series}}{=} \frac{1}{1-\left(\frac{x}{7}\right)} = \frac{7}{7-x}$$

for all $|x| < 7$.

5) Let $f_n : [0, +\infty) \to \mathbb{R}$ be given by $f_n(x) = \dfrac{x^n}{n + x^n}$.

a) Does f_n converge pointwise to some $f(x)$? Where?

b) ———"——— uniformly ———"———? Where?

c) Compute $\lim_{n\to\infty} \int_0^1 f_n(x)\,dx$.

a) $$\lim_{n\to\infty} \frac{x^n}{n + x^n} = \begin{cases} 0 & \text{if } 0 \le x \le 1 \\ 1 & \text{if } x > 1 \end{cases}$$

b/c $$\lim_{n\to\infty} x^n = \begin{cases} 0 & \text{if } 0 \le |x| < 1 \\ 1 & \text{if } x = 1 \\ +\infty & \text{if } x > 1 \\ \text{Does not converge} & \text{if } x < -1 \end{cases}$$

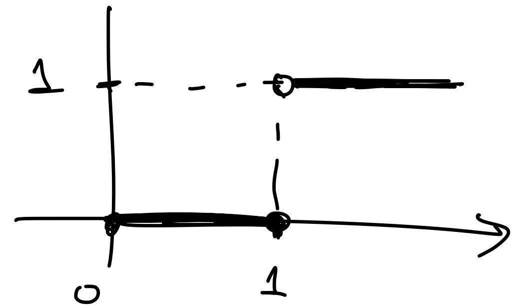
and $$\frac{x^n}{n+x^n} = \frac{1}{\frac{n}{x^n}+1} \longrightarrow 1$$

$\frac{n}{x^n} \xrightarrow[n\to\infty]{} 0$

if $x>1$

a) $f_n(x)$ converges pointwise to

$$f(x) = \begin{cases} 0 & \text{if } 0 \le x \le 1 \\ 1 & \text{if } x > 1. \end{cases}$$



b) No, if the domain contains points $x > 1$ and also points $x \in [0,1]$.

Recall: If $f_n \to f$ uniformly and f_n continuous, then f is continuous.

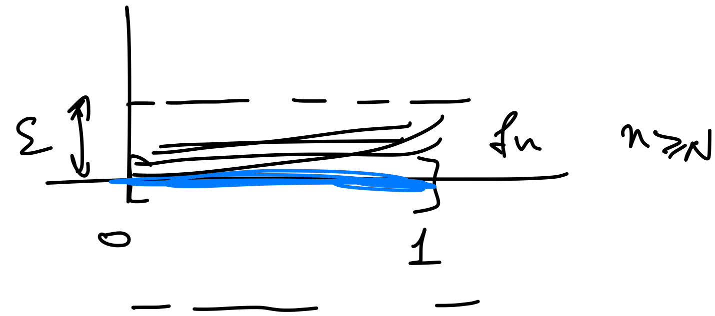
Yes, on the interval $[0,1]$:

$$|f_n(x)| = \left|\frac{x^n}{n+x^n}\right| \le \frac{1}{n}$$ for all $x \in [0,1]$ and $n \in \mathbb{N}$

b/c $$x^n \le 1 \le 1 + \frac{x^n}{n}$$

$\forall \varepsilon > 0 \quad \exists N \in \mathbb{N} \quad \text{s.t.} \quad |f_n(x) - f(x)| < \varepsilon \quad \forall x \in [0,1]$

take $N = \lceil \frac{1}{\varepsilon} \rceil + 1$



uniform convergence

c) $$\lim_{n\to\infty} \int_0^1 f_n(x)\,dx \overset{\text{uniform convergence}}{=} \int_0^1 \underbrace{\lim_{n\to\infty} f_n(x)}_{=0}\,dx$$

$$= \int_0^1 0\,dx = 0.$$