__Weierstrass Approximation Theorem.__ Every continuous function $f:[a,b]\to\mathbb{R}$ can be uniformly approximated by polynomials on $[a,b]$; i.e., there exists a sequence $(B_n f)_{n\in\mathbb{N}}$ of polynomials s.t.

$$\underbrace{B_n f}_{\text{polynomials!}} \longrightarrow f \quad \text{uniformly on } [a,b].$$

just needs to be continuous (need not be differentiable!)

Cf. __Taylor's Theorem:__

$$f:[a,b]\to\mathbb{R},\ x_0\in(a,b) \qquad \Longrightarrow \qquad f(x) \overset{(x_0)}{=} \underbrace{\sum_{n=0}^{+\infty} \frac{f^{(n)}(x_0)}{n!}(x-x_0)^n}_{\text{Taylor Series}}$$

$$\underbrace{|f^{(n)}(x)| \overset{(*)}{\leq} M}_{\text{Much stronger hypothesis! Infinitely many derivatives and all uniformly bounded!}}$$

Say w/ radius of convergence $R>0$

$$T_n(x) = \underbrace{\sum_{k=0}^{n} \frac{f^{(k)}(x_0)}{k!}(x-x_0)^k}_{\text{Taylor polynomial of degree } n \text{ "at } x_0\text{"}};$$

Then $T_n \to f$ uniformly on $[x_0-R_1, x_0+R_1]$ for any $0<R_1<R$

Note: 1. $T_n(x_0) = f(x_0)$, i.e. the approximating Taylor polynomials (of any degree) agree with f at $x = x_0$.

$$2.\ T_{n+1}(x) = \underbrace{T_n(x)}_{\text{fixed}} + \underbrace{\frac{f^{(n+1)}(x_0)}{(n+1)!}(x-x_0)^{n+1}}_{\text{add one more power of } x \text{ without changing any of the lower order terms}}$$

Rmk: Neither of the above will be true in the polynomials $B_n f$ that originate from the Weierstrass Approximation.

Recall: Binomial Theorem $\forall a, b \in \mathbb{R}$, $n \in \mathbb{N}$

$$(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^k b^{n-k}$$

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}$$ "n choose k"

Lemma 1 For every $x \in \mathbb{R}$ and $n \geq 0$ integer,

$$\sum_{k=0}^{n} \binom{n}{k} x^k (1-x)^{n-k} = 1.$$

Pf: Use Binomial Thm w/ $a = x$ and $b = 1-x$, and note that $a + b = x + (1-x) = 1$.

Lemma 2. For every $x \in \mathbb{R}$ and $n \geq 0$ integer,

$$\sum_{k=0}^{n} (nx-k)^2 \binom{n}{k} x^k (1-x)^{n-k} = nx(1-x) \leq \frac{n}{4}$$

Pf: First, note $\quad k! = k(k-1)! \qquad n! = n(n-1)!$

$$k \cdot \binom{n}{k} = k \cdot \frac{n!}{k!(n-k)!} = \frac{n!}{(k-1)!(n-k)!} = \frac{n(n-1)!}{(k-1)!\underbrace{(n-k)!}_{=((n-1)-(k-1))!}} = n\binom{n-1}{k-1}$$

for all $k \geq 1$.

Together with the Binomial Theorem,

$$\sum_{k=0}^{n} \underbrace{k\binom{n}{k}}_{=n\binom{n-1}{k-1}} x^k (1-x)^{n-k} = n \sum_{k=1}^{n} \binom{n-1}{k-1} x^k (1-x)^{n-k}$$

$j = k-1$

$$= n \sum_{j=0}^{n-1} \binom{n-1}{j} x^{j+1} (1-x)^{n-1-j}$$

$$= n \cdot x \underbrace{\sum_{j=0}^{n-1} \binom{n-1}{j} x^j (1-x)^{n-1-j}}_{=1}$$

$$= n \cdot x$$

Similarly, $k(k-1)\binom{n}{k} = n(n-1)\binom{n-2}{k-2}$ for $k \geq 2$

$$\sum_{k=0}^{n} \underbrace{k(k-1)\binom{n}{k}}_{\substack{=n(n-1)\binom{n-2}{k-2} \\ \text{if } k\geq 2}} x^k (1-x)^{n-k} \underset{j=k-2}{=} \sum_{j=0}^{n-2} n(n-1)\binom{n-2}{k-2} x^{j+2} (1-x)^{n-j-2}$$

$$= n(n-1)x^2 \underbrace{\sum_{j=0}^{n-2} \binom{n-2}{k-2} x^j (1-x)^{n-2-j}}_{=1}$$

$$= n(n-1)x^2.$$

Altogether, we have:

$$\sum_{k=0}^{n} k\binom{n}{k} x^k (1-x)^{n-k} = n\cdot x$$

$$\sum_{k=0}^{n} k(k-1)\binom{n}{k} x^k (1-x)^{n-k} = n(n-1)x^2$$

Adding them, we find

$$\sum_{k=0}^{n} k^2 \binom{n}{k} x^k (1-x)^{n-k} = nx + n(n-1)x^2 = nx + n^2x^2 - nx^2$$
$$= n^2x^2 + nx(1-x)$$

So, finally,

$$\sum_{k=0}^{n} \underbrace{(nx-k)^2}_{= n^2x^2 - 2nxk + k^2} \binom{n}{k} x^k (1-x)^{n-k} =$$

$$= \sum_{k=0}^{n} n^2x^2 \binom{n}{k} x^k (1-x)^{n-k} + \sum_{k=0}^{n} (-2nxk) \binom{n}{k} x^k (1-x)^{n-k}$$

$$+ \sum_{k=0}^{n} k^2 \binom{n}{k} x^k (1-x)^{n-k}$$

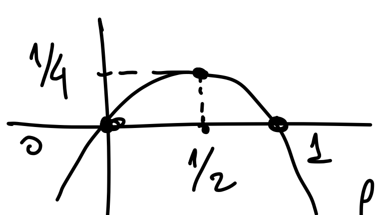
$$= n^2x^2 \underbrace{\left(\sum_{k=0}^{n} \binom{n}{k} x^k (1-x)^{n-k} \right)}_{=1} - 2nx \underbrace{\left(\sum_{k=0}^{n} k \binom{n}{k} x^k (1-x)^{n-k} \right)}_{n \cdot x}$$

$$+ n^2x^2 + nx(1-x)$$

$$= \cancel{n^2x^2} - \cancel{2n^2x^2} + \cancel{n^2x^2} + nx(1-x) = nx(1-x).$$

Note that $nx(1-x) \le \frac{n}{4}$ since $x(1-x) \le \frac{1}{4}$

for all $x \in \mathbb{R}$.



$f(x) = x(1-x) \le \frac{1}{4}$.

$\square$

<u>Definition:</u> Given a function $f:[0,1]\to\mathbb{R}$, define

$$\underset{\substack{\uparrow\\ \text{"Bernstein polynomial" of degree } n \text{ associated to } f(x).}}{(B_n f)(x)} = \underbrace{\sum_{k=0}^{n} \underbrace{f\left(\frac{k}{n}\right)\binom{n}{k}}_{\in\mathbb{R}} \cdot \underbrace{x^k(1-x)^{n-k}}_{\text{polynomial of degree } n \text{ in } x.}}_{\text{polynomial of degree } n \text{ on } x.}$$

<u>Thm.</u> If $f:[0,1]\to\mathbb{R}$ is continuous, then

$B_n f \longrightarrow f$ uniformly on $[0,1]$.

(This is the Weierstrass Approx. Thm for $[a,b]=[0,1]$. The case of general $[a,b]$ follows easily from this.)

<u>Pf:</u> Since f is continuous, it is bounded:

$$M = \sup\{|f(x)| : x\in[0,1]\} \underset{\text{If } f\neq 0.}{>} 0$$

Given $\varepsilon>0$, there exists $\delta>0$ such that

$$x,y\in[0,1] \text{ and } |x-y|<\delta \implies |f(x)-f(y)|<\frac{\varepsilon}{2}$$

b/c f is uniformly continuous on $[0,1]$. We claim:

$$|B_n f(x)-f(x)|<\varepsilon \text{ for all } x\in[0,1] \text{ if } n>N=\left\lceil\frac{M}{\varepsilon\cdot\delta^2}\right\rceil$$

Clearly

$$f(x) = f(x) \cdot \underbrace{\left(\sum_{k=0}^{n} \binom{n}{k} x^k (1-x)^{n-k} \right)}_{1}$$

$$= \sum_{k=0}^{n} f(x) \binom{n}{k} x^k (1-x)^{n-k}$$

Thus

$$|B_n f(x) - f(x)| = \left| \sum_{k=0}^{n} \left(f\left(\frac{k}{n}\right) - f(x) \right) \binom{n}{k} x^k (1-x)^{n-k} \right|$$

Triangle ineq.

$$\leq \sum_{k=0}^{n} \left| f\left(\frac{k}{n}\right) - f(x) \right| \underbrace{\binom{n}{k} \cdot x^k (1-x)^{n-k}}_{\geq 0}$$

To show the above is $< \varepsilon$, we separate the integers $k = 0, 1, 2, \ldots, n$ into two subsets:

$k \in A$ if $\left| \frac{k}{n} - x \right| < \delta$; $\quad k \in B$ if $\left| \frac{k}{n} - x \right| \geq \delta$

Since $A \cup B = \{0, 1, \ldots, n\}$, $A \cap B = \emptyset$;

$$\sum_{k=0}^{n} (--) = \underbrace{\sum_{k \in A} (---)}_{< \frac{\varepsilon}{2}} + \underbrace{\sum_{k \in B} (---)}_{< \frac{\varepsilon}{2}}.$$

If $k \in A$, we have $\left|\frac{k}{n} - x\right| < \delta$ so

$\left|f\left(\frac{k}{n}\right) - f(x)\right| < \frac{\varepsilon}{2}$ and therefore:

$$\sum_{k \in A} \underbrace{\left|f\left(\frac{k}{n}\right) - f(x)\right|}_{< \varepsilon/2} \binom{n}{k} x^k (1-x)^{n-k} < \frac{\varepsilon}{2} \sum_{k \in A} \binom{n}{k} x^k (1-x)^{n-k}$$

$$\leq \frac{\varepsilon}{2} \underbrace{\sum_{k=0}^{n} \binom{n}{k} x^k (1-x)^{n-k}}_{=1}$$

If $k \in B$, then $\left|\frac{k}{n} - x\right| \geq \delta$ ie $\left|\frac{k - nx}{n}\right| \geq \delta$ and so

$|k - nx| \geq \delta \cdot n$ and squaring both sides:

$$(nx - k)^2 = (k - nx)^2 \geq \delta^2 \cdot n^2$$

i.e. $\quad 1 \leq \frac{1}{\delta^2 n^2} (k - nx)^2$

So,

$$\sum_{k \in B} \underbrace{\left|f\left(\frac{k}{n}\right) - f(x)\right|}_{\leq \left|f\left(\frac{k}{n}\right)\right| + |f(x)| \leq M + M = 2M} \binom{n}{k} x^k (1-x)^{n-k} \leq 2M \cdot \sum_{k \in B} 1 \cdot \binom{n}{k} x^k (1-x)^{n-k}$$

$$\leq \frac{2M}{\delta^2 n^2} \sum_{k \in B} (k - nx)^2 \binom{n}{k} x^k (1-x)^{n-k}$$

$$\leq \frac{2M}{\delta^2 \cdot n^2} \cdot \frac{n}{4} = \frac{M}{2n\delta^2} \leq \frac{\varepsilon}{2}$$

by Lemma 2:

$$\sum_{k=0}^{n} (nx-k)^2 \binom{n}{k} x^k (1-x)^{n-k} = nx(1-x) \leq \frac{n}{4}.$$

This concludes the proof of the claim and hence of the theorem.

$\square$