Recall (from Lecture 8):

<u>Bolzano – Weierstrass Thm:</u> Every bounded sequence $(s_n)_{n\in\mathbb{N}}$ of real numbers has a convergent subsequence.

<u>Q:</u> Is there a version of this fact that applies to sequences $(f_n)_{n\in\mathbb{N}}$ of functions?

<u>A:</u> Yes: Arzelà – Ascoli theorem.

What notion of boundedness do we need?

$(f_n)_{n\in\mathbb{N}}$ $\qquad f_n : [a,b] \to \mathbb{R}$

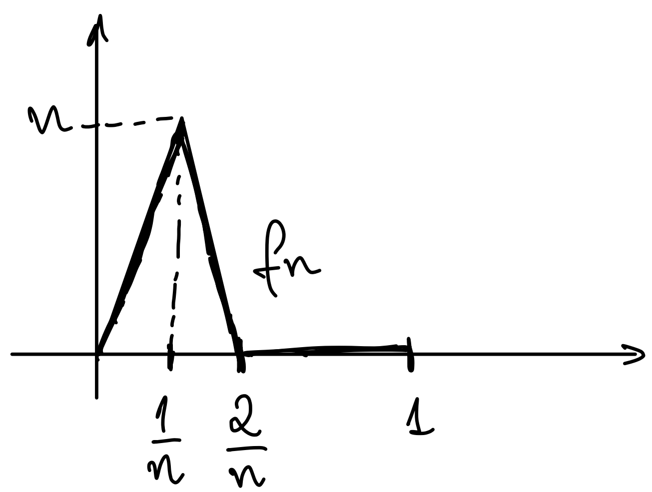
1) <u>pointwise bounded</u> if $\forall n \in \mathbb{N}$ $\exists M_n > 0$ such that $|f_n(x)| \le M_n$, for all $x \in [a,b]$

2) <u>uniformly bounded</u> if $\exists M > 0$ such that $|f_n(x)| \le M$ for all $n \in \mathbb{N}$ and $x \in [a,b]$.

Clearly, 2) $\Rightarrow$ 1). Let us show that 1) $\not\Rightarrow$ 2)

<u>Example:</u> $f_n : [0,1] \to \mathbb{R}$

$$f_n(x) = \begin{cases} n^2 x & \text{if } x \in [0, \frac{1}{n}) \\ n^2(\frac{2}{n} - x) & \text{if } x \in [\frac{1}{n}, \frac{2}{n}] \\ 0 & \text{otherwise} \end{cases}$$



Pointwise bounded ✓

$|f_n(x)| \leq n$

But $(f_n)_{n\in\mathbb{N}}$ is <u>not</u> uniformly bounded $f_n(\frac{1}{n}) = n$

and $\forall M > 0$ $\exists n \in \mathbb{N}$ s.t. $f_n(\frac{1}{n}) = n > M$ hence

$|f_n(x)| \leq M$ is violated for $x = \frac{1}{n}$.

However, even (f_n) is <u>uniformly bounded</u>, that is,

even if $|f_n(x)| \leq M$, $\forall x \in [a,b]$, $n \in \mathbb{N}$, it might

not have any convergent subsequence:

<u>Example:</u> $f_n : [0, 2\pi] \to \mathbb{R}$, $f_n(x) = \sin(nx)$

- Clearly $(f_n)_{n\in\mathbb{N}}$ is uniformly bounded:

$$|f_n(x)| = |\sin(nx)| \leq 1.$$

- But $(f_n)_{n\in\mathbb{N}}$ has no convergent subsequence. (not even pointwise convergent!)

Pf: Suppose, by contradiction, that $f_{n_k}(x) = \sin(n_k x)$ converges pointwise on $[0,2\pi]$. Then for all $x \in [0,2\pi]$ we would have:

$$0 \overset{\circledast}{=} \lim_{k\to\infty} \left(f_{n_k}(x) - f_{n_{k+1}}(x)\right)^2 = \lim_{k\to\infty} \left(\sin(n_k x) - \sin(n_{k+1} x)\right)^2$$

we only saw a special version of this in Lecture 24

By Lebesgue's Dominated Convergence Thm

$$\lim_{k\to\infty} \int_0^{2\pi} \left(\sin(n_k x) - \sin(n_{k+1} x)\right)^2 dx \overset{\circledast}{=} \int_0^{2\pi} 0 \, dx = 0.$$

However, $\int_0^{2\pi} \left(\sin(n_k x) - \sin(n_{k+1} x)\right)^2 dx = 2\pi$, which gives the desired contradiction.

We will prove this in the Arzelà–Ascoli Thm!

An additional property that ensures the existence of convergent subsequences is:

Definition: A family $\mathcal{F} = \{ f_\lambda : [a,b] \to \mathbb{R} : \lambda \in \Lambda \}$ is called equicontinuous if $\forall \varepsilon > 0 \ \exists \delta > 0$ s.t. for all $f_\lambda \in \mathcal{F}$

$$|x-y| < \delta \implies |f_\lambda(x) - f_\lambda(y)| < \varepsilon.$$

Proposition: If $f_n : [a,b] \to \mathbb{R}$ are continuous functions that converge uniformly on $[a,b]$ to $f : [a,b] \to \mathbb{R}$, then $\mathcal{F} = \{ f_n : [a,b] \to \mathbb{R} : n \in \mathbb{N} \}$ is equicontinuous.

Pf: Given $\varepsilon > 0$, since $f_n \to f$ uniformly, $\exists N \in \mathbb{N}$ s.t. if $n \geq N$, then $|f_n(x) - f_N(x)| < \frac{\varepsilon}{3}$ for all $x \in [a,b]$.

Recall continuous function on a closed interval are uniformly continuous, hence, for each $1 \leq i \leq N$, $\exists \delta_i > 0$

s.t. $|x-y| < \delta_i \implies |f_i(x) - f_i(y)| < \frac{\varepsilon}{3}$.

Take $\delta = \min\limits_{1 \leq i \leq N} \delta_i > 0$. Then if $1 \leq i \leq N$ and $|x-y| < \delta$, then $|f_i(x) - f_i(y)| < \frac{\varepsilon}{3}$. If $n \geq N$, then

$$|f_n(x) - f_n(y)| \underset{\text{Triangle ineq.}}{\leq} \underbrace{|f_n(x) - f_N(x)|}_{< \frac{\varepsilon}{3}} + \underbrace{|f_N(x) - f_N(y)|}_{< \frac{\varepsilon}{3}} + \underbrace{|f_N(y) - f_n(y)|}_{< \frac{\varepsilon}{3}}$$

$$< \varepsilon.$$

Altogether, $|x-y| < \delta \implies |f_n(x) - f_n(y)| < \varepsilon$ for all $n \in \mathbb{N}$, i.e., $\mathcal{F} = \{f_n\}$ is equicontinuous. $\square$

Arzelà-Ascoli Theorem. Let $f_n : [a,b] \to \mathbb{R}$ be a sequence of functions which is uniformly bounded and equicontinuous. Then there exists a subsequence f_{n_k} that converges uniformly on $[a,b]$ to $f : [a,b] \to \mathbb{R}$.

Pf: Let $D = \{d_1, d_2, d_3, \dots\} \subset [a,b]$ be a dense and countable subset, e.g., $D = \mathbb{Q} \cap [a,b]$. Since (f_n) is uniformly bounded, say $|f_n(x)| \leq M$ for all $x \in [a,b]$ and $n \in \mathbb{N}$, all of the following are bounded sequences:

$$\left(f_n(d_1)\right)_{n\in\mathbb{N}}, \quad \left(f_n(d_2)\right)_{n\in\mathbb{N}}, \quad \dots$$

By the Bolzano-Weierstrass thm, each of the above (bounded) sequences $|f_n(d_j)| \leq M$, has a convergent subsequence. Let $\left(f_{1,k}(d_1)\right)_{k\in\mathbb{N}}$ be the convergent subsequence of $\left(f_n(d_1)\right)_{n\in\mathbb{N}}$, and let y_1 be its limit:

$$f_{1,k}(d_1) \xrightarrow{k\to\infty} y_1.$$

Note that $f_{1,k}(d_2)$ is also bounded, so there is a subsequence $f_{2,k}$ such that converges:

$$f_{2,k}(d_2) \xrightarrow{k\to\infty} y_2$$

Continuing in this way, taking subsequences, we get:

$$\begin{array}{lllllll}
 & f_n & : & f_1, & f_2, & f_3, & \cdots \\
\text{subseq.} & & & & & & \\
\boxed{m=1} & f_{1,k} & : & f_{1,1}, & f_{1,2}, & f_{1,3}, & \cdots \\
\text{subseq.} & & & & & & \\
\boxed{m=2} & f_{2,k} & : & f_{2,1}, & f_{2,2}, & f_{2,3} & \cdots \\
\text{subseq.} & & & & & & \\
\boxed{m=3} & f_{3,k} & : & f_{3,1}, & f_{3,2}, & f_{3,3} & \cdots \\
 & \vdots & & \vdots & \vdots & \vdots &
\end{array}$$

such that $(f_{m,k})_{k \in \mathbb{N}}$ is a subsequence of $(f_{m-1,k})_{k \in \mathbb{N}}$; if $j \le m$ then $f_{m,k}(d_j) \longrightarrow y_j$ as $k \to \infty$.

Choose $k(m) \geq m$ large enough so that if

$$j \le m, \quad k(m) \le k \implies |f_{m,k}(d_j) - y_j| < \frac{1}{m}.$$

The "superdiagonal" subsequence

$$g_m(x) := f_{m,k(m)}(x)$$

converges to a limit at every $d_j \in D$, namely

$$g_m(d_j) \xrightarrow{m \to \infty} y_j$$

We claim that $(g_m)_{m\in\mathbb{N}}$ converges uniformly on $[a,b]$. From results in Lecture 17, it is enough to show that $(g_m)_{m\in\mathbb{N}}$ is uniformly Cauchy.

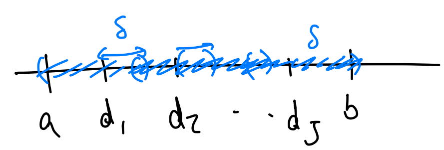
$$\boxed{\begin{array}{l}\forall \varepsilon>0\ \exists N\in\mathbb{N} \text{ s.t. } n,m\geq N \\ |g_n(x)-g_m(x)|<\varepsilon,\ \forall x\in[a,b]\end{array}}$$

Since $\mathcal{F}=\{f_n\}$ is equicontinuous, also $\mathcal{G}=\{g_m\}$ is equicontinuous (because $\mathcal{G}\subset\mathcal{F}$). So, given $\varepsilon>0$, there exists $\delta>0$ s.t. for all $x,y\in[a,b]$, $m\in\mathbb{N}$,

$$|x-y|<\delta \overset{①}{\Longrightarrow} |g_m(x)-g_m(y)|<\frac{\varepsilon}{3}$$

Choose $J\in\mathbb{N}$ large enough so that $\{d_1,d_2,\dots,d_J\}$ is such that $\forall x\in[a,b]$ $\exists\, 1\leq j\leq J$ with

$$|x-d_j|<\delta.$$



Every $(g_m(d_j))_{m\in\mathbb{N}}$ is convergent (to y_j) and hence Cauchy, i.e. $\exists N_j\in\mathbb{N}$ s.t. $n,m\geq N_j$ then

$$|g_n(d_j)-g_m(d_j)|\overset{②}{<}\frac{\varepsilon}{3}.$$

Let $N = \max_{1 \le j \le J} N_j$. Then, if $n, m \ge N$,

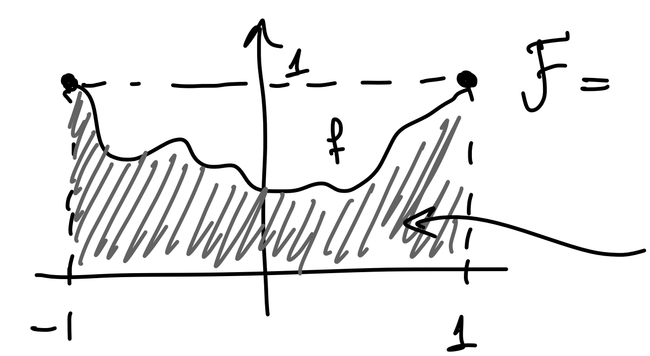
$$|g_n(x) - g_m(x)| \overset{\text{triangle ineq.}}{\le} \underbrace{|g_n(x) - g_n(d_j)|}_{< \varepsilon/3 \quad 1} + \underbrace{|g_n(d_j) - g_m(d_j)|}_{< \varepsilon/3 \quad 2}$$

$$+ \underbrace{|g_m(d_j) - g_m(x)|}_{< \varepsilon/3 \quad 1} < \varepsilon.$$

where d_j, $1 \le j \le J$ is such that $|x - d_j| < \delta$.

Thus $(g_n)_m$ is uniformly Cauchy and hence it has a uniformly convergent subsequence. $\square$

<u>What is the least area?</u> "Calculus of Variations"



$$\mathcal{F} = \left\{ f : [-1,1] \to [0,1], \text{ continuous s.t. } f(-1) = 1 = f(1) \right\}$$

$$A(f) = \int_{-1}^{1} f(x)\,dx \qquad \text{"}\min_{f \in \mathcal{F}} A(f) = ?\text{"}$$

<u>Q</u>. Does there exist $f_0 \in \mathcal{F}$ s.t. $A(f) \ge A(f_0)$ for all $f \in \mathcal{F}$?

A: No: there is no such $f_0 \in \mathcal{F}$.

For any $f \in \mathcal{F}$, $A(f) > 0$. ⊛

$\forall n \in \mathbb{N}$, consider $f_n(x) = x^{2n}$. Clearly $f_n \in \mathcal{F}$.

$$A(f_n) = \int_{-1}^{1} x^{2n}\, dx = \frac{x^{2n+1}}{2n+1}\Big|_{-1}^{1} = \frac{2}{2n+1} \xrightarrow{n \to +\infty} 0.$$

So if $f_0 \in \mathcal{F}$ existed, $A(f_0) \leq A(f_n) = \dfrac{2}{2n+1}$

so $A(f_0) = 0$. This contradiction implies that no such $f_0 \in \mathcal{F}$ exists. ⊛

Rmk: $\mathcal{F}$ is not equicontinuous.

Indeed, if $\mathcal{F}$ was equicontinuous, then $\{f_n(x) = x^{2n}\}$ would also be equicontinuous; and hence by Arzelà–Ascoli, it would have a unif. conv. subsequence.

Q': Is there a fix?

A': Yes: consider the following subclass of $\mathcal{F}$:

$$\mathcal{F}_c = \left\{ f:[-1,1]\to[0,1] \;:\; f(-1)=1=f(1),\ \underbrace{|f(x)-f(y)|\le c|x-y|,\ \forall x,y\in[-1,1]}_{\text{Lipschitz w/ const. } c} \right\}$$

Note: $\forall c>0$, $\mathcal{F}_c \subsetneq \mathcal{F}$

Claim: $\mathcal{F}_c$ is equicontinuous.

$\forall \varepsilon>0$, let $\delta = \varepsilon/c$. Then if $x,y\in[-1,1]$

$$|x-y|<\delta \implies |f(x)-f(y)| \le c|x-y| < c\cdot\delta = \varepsilon.$$

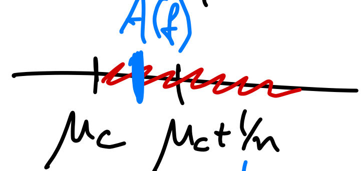
Clearly, $\mathcal{F}_c$ is uniformly bounded:

$$|f(x)| \le 1 \quad \forall x\in[-1,1], \quad \forall f\in\mathcal{F}_c$$

By Arzelà–Ascoli, any sequence of functions in $\mathcal{F}_c$ has a subsequence that converges uniformly on $[-1,1]$.

Let $\mu_c = \inf\{A(f) : f\in\mathcal{F}_c\}$

- non-empty
- bounded from below by 0.



$\forall n\in\mathbb{N}$, $\exists f_n\in\mathcal{F}_c$ s.t.

$$\mu_c \le A(f_n) \le \mu_c + \frac{1}{n} \quad (*)$$

Let $\{f_{n_k}\}$ be a subsequence of $\{f_n\}$ that converges uniformly; say $f_{n_k} \longrightarrow \phi_c : [-1,1] \to [0,1]$.

$$A(\phi_c) = \int_{-1}^{1} \phi_c(x)\,dx = \int_{-1}^{1} \lim_{k\to\infty} f_{n_k}(x)\,dx = \lim_{k\to\infty} \int_{-1}^{1} f_{n_k}(x)\,dx$$
$$= \lim_{k\to\infty} A(f_{n_k}) \overset{(*)}{=} \mu_c.$$

So we found a continuous function $\phi_c : [-1,1] \to [0,1]$ with $\phi_c(-1) = 1 = \phi_c(1)$ which <u>attains</u> the inf.; i.e., the "area under" ϕ_c is the least possible among the areas under functions in $\tilde{\mathcal{F}}_c$.

<u>Rmk</u>: What does $\phi_c : [-1,1] \to [0,1)$ look like?