Taylor Series

From past lectures: we may define functions via power series:

$$f : (-R, R) \longrightarrow \mathbb{R}$$

$R = \frac{1}{\beta}$, $\beta = \limsup_{n\to\infty} |a_n|^{1/n}$

"radius of convergence"

$$f(x) = \sum_{n=0}^{+\infty} a_n x^n$$

$(x-x_0)^n$, $x_0 = 0$

"centered at $x_0 = 0$"

Q: What's the relation between $(a_n)_{n\geq 0}$ and $f(x)$?

"A:" From Lecture 18, we may differentiate term-by-term:

$k=0$: $$f(x) = \sum_{n=0}^{+\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + \dots$$

$k=1$: $$f'(x) = \sum_{n=1}^{+\infty} n \cdot a_n \cdot x^{n-1} = a_1 + 2a_2 x + 3a_3 x^2 + 4a_4 x^3 + \dots$$

$k=2$: $$f''(x) = \sum_{n=2}^{+\infty} n \cdot (n-1) a_n \cdot x^{n-2} = 2a_2 + 3\cdot 2 a_3 x + 4 \cdot 3 \cdot a_4 x^2 + \dots$$

By induction, we arrive at:

k: $$f^{(k)}(x) = \sum_{n=k}^{+\infty} n \cdot (n-1) \cdot \ldots \cdot (n-k+1) \cdot a_n \cdot x^{n-k}$$

If we compute the above at $x=0$, the result is the term with $n=k$, since all other terms vanish:

$$f^{(n)}(0) = f^{(k)}(0) = n\cdot(n-1)\cdots(\underbrace{n-\cancelto{n}{k}+1}_{1})\cdot a_n = n!\cdot a_n$$

Thus: $f^{(n)}(0) = n!\cdot a_n$ i.e. $a_n = \dfrac{f^{(n)}(\overset{x_0=0}{0})}{n!}$

This motivates defining the following

<u>Def</u>: The Taylor Series of $f(x)$ centered at $x=x_0$ is $\displaystyle\sum_{n=0}^{+\infty} \frac{f^{(n)}(x_0)}{n!}(x-x_0)^n$. The <u>remainder</u> of order n for this Taylor series is

$$R_n(x) = f(x) - \sum_{k=0}^{n-1} \frac{f^{(k)}(x_0)}{k!}(x-x_0)^k.$$

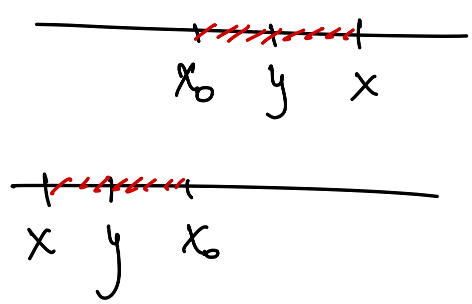
Thus

$$\underbrace{f(x) = \sum_{k=0}^{+\infty} \frac{f^{(k)}(x_0)}{k!}(x-x_0)^k}_{\text{The function } f \text{ is equal to its Taylor Series at } x.} \iff \lim_{n\to\infty} R_n(x) = 0$$

Taylor's Thm. Let $f:(a,b)\to\mathbb{R}$, where $-\infty \le a < b \le +\infty$. Suppose the n^{th} derivative $f^{(n)}$ exists for all $x\in(a,b)$. Then, for all $x\neq x_0$, there exists y between x and x_0 s.t.:

$$R_n(x) = \frac{f^{(n)}(y)}{n!}(x-x_0)^n$$

Remainder of order n for the Taylor Series of $f(x)$ centered at $x=x_0$



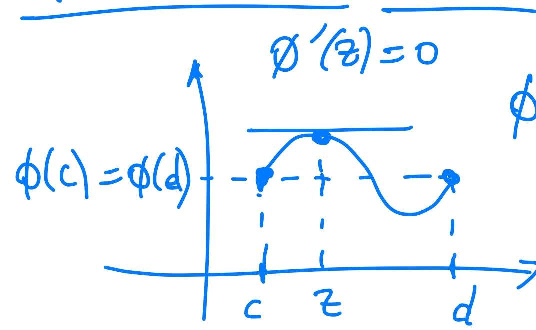
Proof. Fix $x\neq x_0$ and $n\geq 1$. Let M be the solution to

$$f(x) = \sum_{k=0}^{n-1}\frac{f^{(k)}(x_0)}{k!}(x-x_0)^k + \frac{M\cdot(x-x_0)^n}{n!} \qquad (*)$$

$$\left(\text{i.e.,}\quad M = \frac{n!}{(x-x_0)^n}\left(f(x) - \sum_{k=0}^{n-1}\frac{f^{(k)}(x_0)}{k!}(x-x_0)^k\right).\right)$$

Then it suffices to show that $f^{(n)}(y)=M$ for some y between x_0 and x.

Rolle's Thm (Lecture 20)

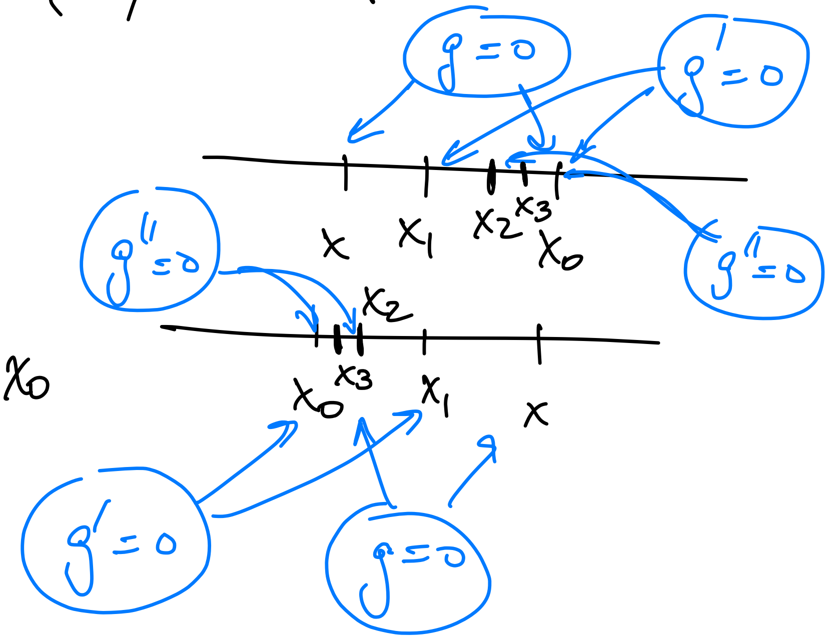


Define $g:(a,b)\to\mathbb{R}$ as follows:

$$g(t) = \sum_{k=0}^{n-1} \frac{f^{(k)}(x_0)}{k!}(t-x_0)^k + \frac{M(t-x_0)^n}{n!} - f(t)$$

Note $g(x_0) = 0$ and $g^{(k)}(x_0) = 0$ for all $k \leq n-1$.

Moreover $g(x) = 0$ by (*).



By Rolle's Thm, there exists x_1 between x and x_0 such that $g'(x_1) = 0$.

Since g' vanishes at $t = x_1$ and $t = x_0$, by Rolle's Thm (again), there exists x_2 between x_0 and x_1 such that $g''(x_2) = 0$. Repeated applications of Rolle's Thm produce a finite sequence $x_1, x_2, x_3, x_4, \ldots, x_n$ such that $g^{(k)}(x_k) = 0$.

At $k = n$, we get $g^{(n)}(x_n) = 0$. But we have that $g^{(n)}(t) = M - f^{(n)}(t)$, so vanishing at $t = x_n$ means that $0 = g^{(n)}(x_n) = M - f^{(n)}(x_n)$, i.e., $f^{(n)}(x_n) = M$. Take $y = x_n$.

$\square$

<u>Corollary</u>: Let $f : (a,b) \to \mathbb{R}$, where $-\infty \le a < b \le +\infty$.

Suppose all derivatives $f^{(n)}$ exist, for all $n \geq 1$ and $x \in (a,b)$, and

$|f^{(n)}(x)| \le C$ for all $n \ge 1$ and $x \in (a,b)$. Then

$$\lim_{n\to\infty} \underbrace{R_n(x)}_{} = 0 \quad \text{for all } x \in (a,b).$$

Remainder of order n
in Taylor Series of $f(x)$
centered at $x_0 \in (a,b)$

In particular, in this case, $f(x) = \sum_{k=0}^{+\infty} \frac{f^{(k)}(x_0)}{k!}(x-x_0)^k$.

<u>Proof</u>: By Taylor's Thm, we have that $\forall n \geq 1$ $\exists y_n$ between x and x_0 such that $R_n(x) = \frac{f^{(n)}(y_n)}{n!}(x-x_0)^n$. Since $|f^{(n)}(x)| \le C$, we have

$$0 \le |R_n(x)| = \frac{|f^{(n)}(y_n)|}{n!}|x-x_0|^n \le \frac{C}{n!}|x-x_0|^n \xrightarrow{n\to\infty} 0$$

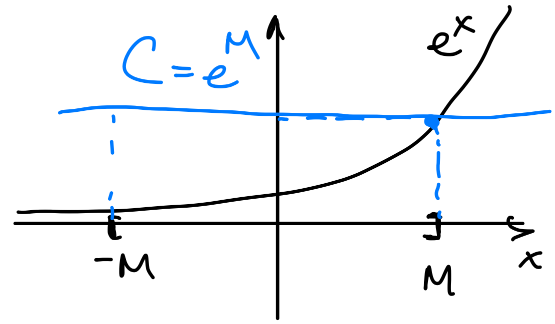
since $\lim_{n\to\infty} \frac{|x-x_0|^n}{n!} = 0$, and hence $\lim_{n\to\infty} R_n(x) = 0$.

$\square$

This Corollary allows us to find many examples of functions that coincide with their Taylor Series.

__Example.__ $f:\mathbb{R}\to\mathbb{R}$, $f(x)=e^x$

From Calculus, we know $f^{(n)}(x)=f(x)=e^x$; so on any fixed interval $(-M,M)$, $M>0$, all derivatives of $f(x)$ are bounded, namely $|f^{(n)}(x)|=|e^x|\le e^M$.



Taking $C=e^M$ in the previous Corollary, we conclude that $f(x)=e^x$ agrees with its Taylor Series at all points in $(-M,M)$. Since $M>0$ is arbitrary, it follows that the same conclusion holds on all $\mathbb{R}$.

__Recall:__ Taylor Series of $f(x)=e^x$ centered at $x_0=0$ is $\displaystyle\sum_{n=0}^{+\infty}\frac{x^n}{n!}$

Thus,
$$\underbrace{e^x}_{f(x)} \overset{\text{(From above Corollary)}}{=} \underbrace{\sum_{n=0}^{+\infty}\frac{x^n}{n!}}_{\text{Taylor Series of } f(x) \text{ centered at } x=x_0}$$

Similarly, you can use the Corollary above to show

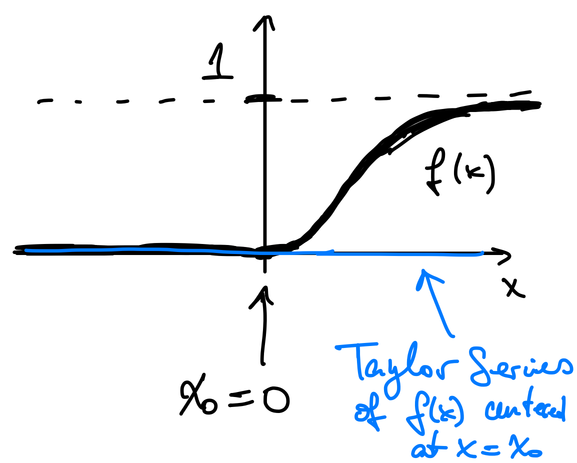
$$\sin x \overset{\left(\text{Use } C=1\right)}{=} \underbrace{\sum_{k=0}^{+\infty} \frac{(-1)^k}{(2k+1)!}\, x^{2k+1}}_{\text{Taylor Series of } \sin x \text{ centered at } x=0}$$

$$\cos x \overset{\left(\text{Use } C=1\right)}{=} \underbrace{\sum_{k=0}^{+\infty} \frac{(-1)^k}{(2k)!}\, x^{2k}}_{\text{Taylor Series of } \cos x \text{ centered at } x=0.}$$

Example of a function that DOES NOT agree with its Taylor Series.

Consider $f : \mathbb{R} \to \mathbb{R}$ defined as

$$f(x) = \begin{cases} e^{-1/x} & \text{if } x > 0 \\ 0 & \text{if } x \le 0 \end{cases}$$



$$f(x) \neq \underbrace{0}_{\text{Taylor Series of } f(x) \text{ centered at } x_0=0.}$$

Claim: Taylor Series of $f(x)$ at $x_0 = 0$ vanishes identically.

- Since $f \equiv 0$ for $x \le 0$, all limits defining derivatives of f vanish when taken from the left $(x \to 0_-)$
- Let us then focus on lateral limits from the right $x \to 0_+$

$$e^{-1/x} \to 0, \quad -1/x \to -\infty$$

$k=0;$ $$\lim_{x\to 0_+} e^{-1/x} = 0.$$

$$\boxed{f(x_0) = 0} \longleftarrow$$ 0^{th} order term in Taylor Series

$k=1;$ $$\frac{d}{dx}\left(e^{-1/x}\right) = e^{-1/x} \cdot \frac{d}{dx}\left(-\frac{1}{x}\right) = e^{-1/x} \cdot \frac{1}{x^2}$$

$$\lim_{x\to 0_+} \underbrace{\frac{1}{x^2}}_{\to +\infty}\ \underbrace{e^{-1/x}}_{\to 0} = 0$$

L'Hospital: Substitute $y = \frac{1}{x} \xrightarrow{x\to 0_+} +\infty$

$$\frac{e^{-1/x}}{x^2} = \frac{e^{-y}}{1/y^2} = y^2 e^{-y} = \frac{y^2}{e^y} \xrightarrow{y\to+\infty} 0$$

$$\lim_{y\to\infty} \frac{y^2}{e^y} \overset{L.H.}{=} \lim_{y\to\infty} \frac{2y}{e^y} \overset{L.H}{=} \lim_{y\to\infty} \frac{2}{e^y} = 0$$

$$\boxed{f'(x_0) = 0} \longleftarrow$$ 1^{st} order term in Taylor Series

$k=2;$ $$\frac{d^2}{dx^2}\left(e^{-1/x}\right) = \frac{d}{dx}\left(\frac{e^{-1/x}}{x^2}\right) = \frac{e^{-1/x}\cdot\frac{1}{\cancel{x^2}}\cdot\cancel{x^2} - e^{-1/x}\cdot 2x}{x^4}$$

$$= \frac{1}{x^4}e^{-1/x} - \frac{2}{x^3}e^{-1/x} = \left(\frac{1}{x^4} - \frac{2}{x^3}\right)e^{-1/x}.$$

$$\lim_{x\to 0_+} \underbrace{\left(\frac{1}{x^4} - \frac{2}{x^3}\right)}_{\text{polynomial in } \frac{1}{x}} \underbrace{e^{-1/x}}_{\text{goes to zero faster.}} = 0$$

L'Hospital

$$\boxed{\frac{f''(x_0)}{2!} = 0} \leftarrow$$

2nd term in Taylor Series

$\vdots$

k

$$\left(\frac{d}{dx}\right)^k \left(e^{-1/x}\right) = \underbrace{\left(\; - - - \; \right)}_{\text{polynomial in } 1/x} e^{-\frac{1}{x}}$$

goes to zero faster.

$$\lim_{x\to 0_+} \left(\frac{d}{dx}\right)^k e^{-1/x} = 0.$$

$$\Downarrow$$

$$\boxed{\frac{f^{(k)}(x_0)}{k!} = 0} \leftarrow$$

kth term in Taylor Series

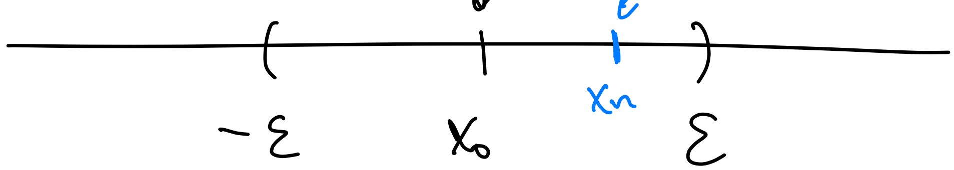
Rigorous proof can be made using induction on k, see Example 3 in §31 of Ross.

<u>Rmk:</u> The derivatives $f^{(n)}(x)$ of the above $f(x)$ are not <u>uniformly bounded</u> in any open neighborhood of $x_0 = 0$, i.e., $\nexists\, C \in \mathbb{R}$ s.t.

$$|f^{(n)}(x)| \leq C \quad \text{for all } n \geq 1 \text{ and } x \in (-\varepsilon, \varepsilon).$$

$f^{(n)}(0) = 0$

For $n \to \infty$, there exist $|x_n| < \varepsilon$ s.t. $f^{(n)}(x_n) \to +\infty$.



Not only the Corollary does not apply, the conclusion actually fails, as we've seen above.