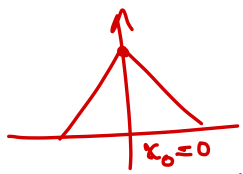
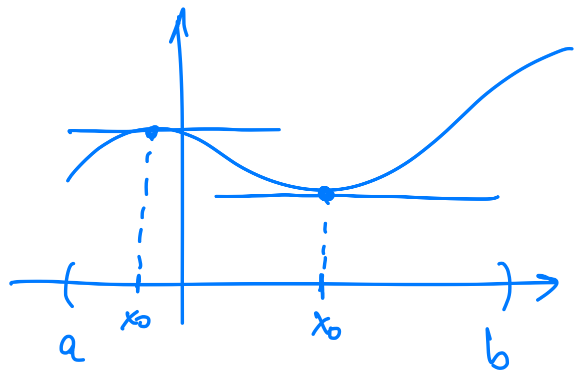
Theorem. Suppose $f : I \subset \mathbb{R} \to \mathbb{R}$ is defined on an open interval $I = (a,b)$ containing x_0, and assumes a maximum (or minimum) at $x = x_0$. If $f(x)$ is differentiable at $x = x_0$, then $f'(x_0) = 0$.

e.g. $f(x) = 1 - |x|$



has a max. at $x = x_0 = 0$ but is not diff. at $x = x_0$.

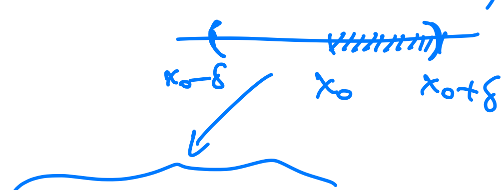


Proof: Say $f(x)$ assumes a maximum at $x = x_0$. Suppose by contradiction that $f'(x_0) \neq 0$.

If $f'(x_0) > 0$, then

$$0 < f'(x_0) = \lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0}$$

$x \in (x_0 - \delta, x_0 + \delta)$



there exists $\delta > 0$ such that if $0 < |x - x_0| < \delta$ then

$$\frac{f(x) - f(x_0)}{x - x_0} > 0.$$ If we take $x_0 < x < x_0 + \delta$,

then $x - x_0 > 0$ and $f(x) \leq f(x_0)$ since $f(x_0)$ is a max.

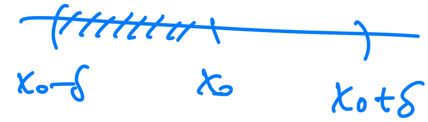
This contradicts $\dfrac{\overbrace{f(x)-f(x_0)}^{>0 \rightsquigarrow f(x)>f(x_0)}}{\underbrace{x-x_0}_{>0}} > 0$.

If $f'(x_0) < 0$, then proceed as above and find $\delta > 0$ such that if $0 < |x - x_0| < \delta$, then

$$\frac{f(x)-f(x_0)}{x-x_0} < 0.$$

If we take $x_0 - \delta < x < x_0$, then

$x - x_0 < 0$ so $f(x) - f(x_0) > 0$

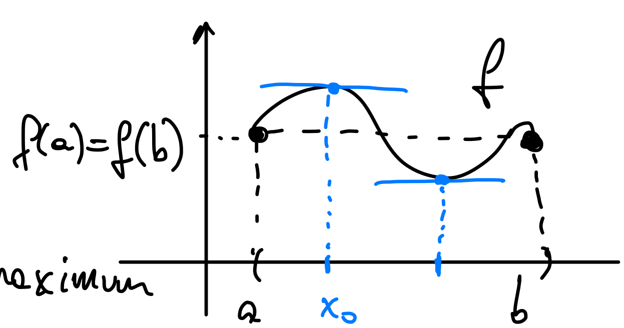


i.e. $f(x) > f(x_0)$, contradicting the fact that $f(x)$ assumes a maximum at $x = x_0$, so $f(x) \leq f(x_0)$.

Therefore $f'(x_0) = 0$ as desired. $\square$

<u>Rolle's Theorem</u>. Let $f : [a,b] \to \mathbb{R}$ be a continuous function that is differentiable on (a,b), and s.t. $f(a) = f(b)$. Then there exists $x_0 \in (a,b)$ such that $f'(x_0) = 0$.

<u>Pf:</u> From the Bolzano–Weierstrass Thm, the function $f(x)$ assumes a minimum and a maximum on the closed interval $[a,b]$, say



$x_{min}, x_{max} \in [a,b]$ such that $f(x_{min}) \leq f(x) \leq f(x_{max})$

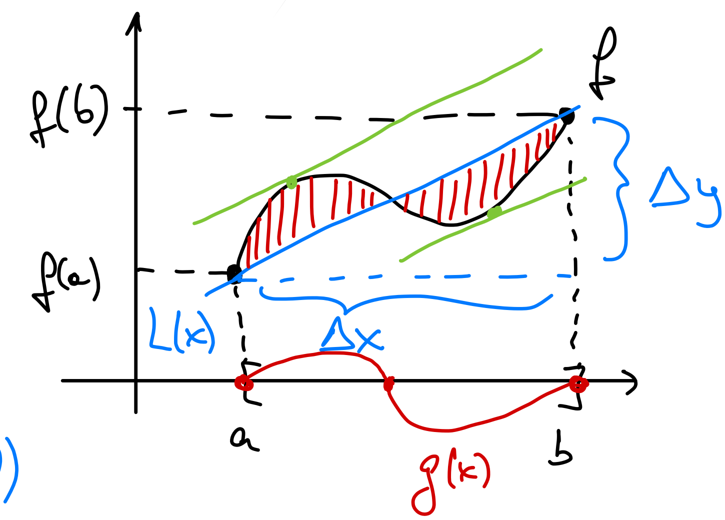
for all $x \in [a,b]$. If x_{min}, x_{max} are both endpoints of $[a,b]$, then $f(x)$ is constant, so $f'(x) \equiv 0$. If not, then at least one among x_{min}, x_{max} is in the open interval (a,b); so by the previous theorem we have that $f'(x_0) = 0$ at that point. $\square$

<u>Mean Value Theorem</u>. Let $f: [a,b] \to \mathbb{R}$ be a continuous function which is differentiable on (a,b). Then there exists $x_0 \in (a,b)$ such that

$$\underbrace{f'(x_0)}_{} = \underbrace{\frac{f(b)-f(a)}{b-a}}_{}$$

slope of tangent line to graph of $f(x)$ at $x = x_0$

slope of the line joining $(a, f(a))$ to $(b, f(b))$



<u>Pf</u>: Apply Rolle's theorem to the auxiliary function

$$g(x) = f(x) - L(x)$$

$$\begin{cases} L(a) = f(a) \\ L(b) = f(b) \end{cases}$$

$$L(x) = \frac{f(b)-f(a)}{b-a} \cdot (x-a) + f(a)$$

Since $g(a) = f(a) - L(a) = 0$

$g(b) = f(b) - L(b) = 0$

and $g : [a,b] \longrightarrow \mathbb{R}$ is continuous and differentiable on (a,b), we may apply Rolle's Theorem to $g(x)$ and conclude that there exists $x_0 \in (a,b)$ such that $g'(x_0) = 0$.

Note
$$g'(x) = f'(x) - L'(x) = f'(x) - \frac{f(b)-f(a)}{b-a}$$

So $g'(x_0) = 0 \Rightarrow f'(x_0) = \dfrac{f(b)-f(a)}{b-a}$. $\square$

<u>Consequences of the Mean Value Theorem:</u>

<u>Prop</u>: If $f : (a,b) \to \mathbb{R}$ is differentiable, then:

(i) f is strictly increasing if $f'(x) > 0$ for all $x \in (a,b)$

(ii) —— " —— decreasing if $f'(x) < 0$ —— " ——

(iii) f is increasing if $f'(x) \geq 0$ for all $x \in (a,b)$

(iv) f is decreasing if $f'(x) \leq 0$ —— " ——

Pf (i) take $a < x_1 < x_2 < b$. By Mean Value Theorem there exists $x_0 \in (x_1, x_2)$ such that

$$0 < f'(x_0) = \frac{f(x_2) - f(x_1)}{\underbrace{x_2 - x_1}_{>0}}$$

so $f(x_2) - f(x_1) > 0$, i.e. $f(x_1) < f(x_2)$.

Similarly for (ii), (iii), and (iv). $\square$

Cor. If $f : (a,b) \to \mathbb{R}$ is differentiable and such that $f'(x) = 0$ for all $x \in (a,b)$, then $f(x)$ is constant on (a,b).

Pf. Suppose $f(x)$ is not constant, i.e. there exist $x_1 < x_2$ such that $f(x_1) \neq f(x_2)$. By the Mean Value Theorem, there exists $x_0 \in (x_1, x_2)$ s.t.

$$0 = f'(x_0) = \frac{\overbrace{f(x_2) - f(x_1)}^{=0}}{\underbrace{x_2 - x_1}_{>0}}$$

so $f(x_1) = f(x_2)$, giving the desired contradiction. $\square$

<u>Cor</u>: Let f and g be differentiable functions on (a,b) such that $f'(x) = g'(x)$ for all $x \in (a,b)$. Then there exists $c \in \mathbb{R}$ such that $f(x) = g(x) + c$; for all $x \in (a,b)$.

<u>Pf</u>: Apply previous corollary to $f(x) - g(x)$. $\square$

<u>Inverse Function</u>

<u>Theorem</u>. Let $f: (a,b) \to \mathbb{R}$ be an injective continuous function. If $f(x)$ is differentiable at $x = x_0 \in (a,b)$ and $f'(x_0) \neq 0$, then $f^{-1}(y)$ is differentiable at $y = y_0 = f(x_0)$ and
$$(f^{-1})'(y_0) = \frac{1}{f'(x_0)}.$$

<u>Pf</u>: Note the image $f((a,b)) = (c,d)$ is an open interval.
$$0 \neq f'(x_0) = \lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0},$$
so we have:
$$\lim_{x \to x_0} \frac{x - x_0}{f(x) - f(x_0)} = \frac{1}{f'(x_0)}.$$
So for any $\varepsilon > 0$, there exists $\delta > 0$ such that $0 < |x - x_0| < \delta$, then

$$\left| \frac{x - x_0}{f(x) - f(x_0)} - \frac{1}{f'(x_0)} \right| < \varepsilon. \quad \text{Let } g = f^{-1}.$$

exists b/c f was assumed injective!

$g(y) = x \iff f(x) = y$

We wish to show $g(y)$ is differentiable at $y = y_0$. By earlier results, g is continuous, so there exists $\eta > 0$ s.t. $0 < |y - y_0| < \eta$ then $|g(y) - \underset{x_0}{\underset{\parallel}{g(y_0)}}| < \delta$, i.e. $|g(y) - x_0| < \delta$. By the above

$$0 < |y - y_0| < \eta \implies \left| \frac{g(y) - x_0}{f(g(y)) - f(x_0)} - \frac{1}{f'(x_0)} \right| < \varepsilon$$

Thus, since $\dfrac{g(y) - x_0}{f(g(y)) - f(x_0)} = \dfrac{g(y) - g(y_0)}{y - y_0}$ we conclude

that $\displaystyle\lim_{y \to y_0} \frac{g(y) - g(y_0)}{y - y_0} = \frac{1}{f'(x_0)}$; so $g(y)$ is diff at $y = y_0$ and $g'(y_0) = \dfrac{1}{f'(x_0)}$. $\square$

Example: $f : [0, +\infty) \to \mathbb{R}$, $f(x) = x^n$, $n \in \mathbb{N}$

is injective and has inverse function given by

$g : [0, +\infty) \to \mathbb{R}$, $g(y) = y^{1/n} = \sqrt[n]{y}$.

By the above Theorem, g is differentiable, and

$$g'(y_0) = \frac{1}{f'(x_0)} = \frac{1}{n x_0^{n-1}} = \frac{1}{n\left(y_0^{1/n}\right)^{n-1}} = \frac{1}{n \cdot y_0^{\frac{n-1}{n}}}$$

$$x_0^n = f(x_0) = y_0$$

$$y_0^{1/n} = g(y_0) = x_0$$

$$= \frac{1}{n} \cdot y_0^{\frac{1-n}{n}} = \frac{1}{n} \cdot y_0^{\frac{1}{n} - 1}$$

So: $\frac{d}{dy}\left(y^{1/n}\right) = \frac{1}{n} y^{\frac{1}{n} - 1}$