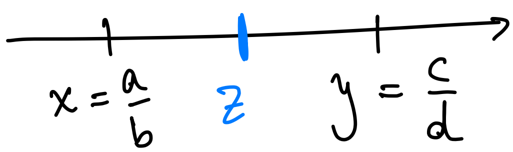
MAT 320/640 Lecture 2 8/30/2021

Rational numbers

$\mathbb{Z} = \{\ldots, -2, -1, 0, 1, 2, \ldots\}$

$$\mathbb{Q} = \left\{ \frac{a}{b} : a, b \in \mathbb{Z}, \quad b \neq 0 \right\}$$

Q1: Given two rational numbers $x, y \in \mathbb{Q}$, is there another rational number $z \in \mathbb{Q}$ s.t. $x < z < y$?



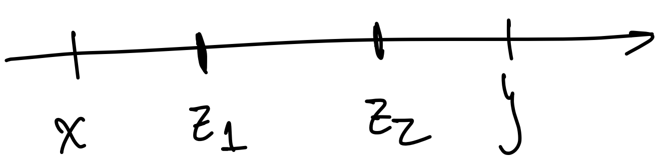
A: Yes, for example, we may take

$$z = \frac{x+y}{2} = \frac{1}{2}\left(\frac{a}{b} + \frac{c}{d}\right) \in \mathbb{Q}$$

$a, b, c, d \in \mathbb{Z}$, $b \neq 0$, $d \neq 0$

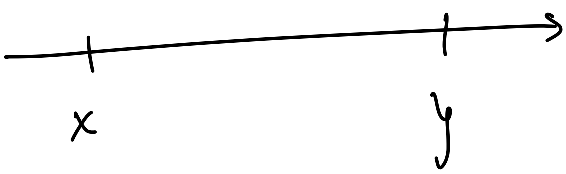
E.g., we can also take weighted averages such as

$$z_1 = \frac{2x+y}{3}, \qquad z_2 = \frac{x+2y}{3} \quad \in \mathbb{Q}$$



Can iterate and get infinitely many rational numbers between any two $x, y \in \mathbb{Q}$.

Q2: Given $x, y \in \mathbb{Q}$, are <u>all</u> numbers between x and y also rational?



(<u>Note</u>: by "all numbers in between" we mean every $z \in \underline{\mathbb{R}}$ such that $x < z < y$) $\boxed{\text{not yet defined}}$

<u>A2</u>: <u>No</u>, for example, consider $x = 1$ and $y = 2$ and $z = \sqrt{2}$. $z^2 = 2$; i.e. $z^2 - 2 = 0$

<u>Prop</u>: $\sqrt{2}$ is not a rational number ($\sqrt{2} \notin \mathbb{Q}$).

<u>Pf</u>: <u>Suppose $\sqrt{2} \in \mathbb{Q}$</u>, that is $\sqrt{2} = \frac{a}{b}$, $a, b \in \mathbb{Z}$, $b \neq 0$.

Proof by contradiction.

simplified fraction: a and b have <u>no common divisors</u>.

Then $b\sqrt{2} = a$, so $2b^2 = a^2$.

Thus a^2 is even, then a is also even.

Then a^2 must be divisible by 4. Thus b^2 is even, because $b^2 = \frac{a^2}{2}$ (div. by 4). So b is also even.

But a and b cannot both be even, because we assumed they had no common divisors. This contradiction proves that $\sqrt{2} \notin \mathbb{Q}$. $\square$

Rational Zeros Theorem

Let $n \in \mathbb{N}$, and $a_n, a_{n-1}, \dots, a_1, a_0 \in \mathbb{Z}$. Consider the equation

(simplified)

$$a_n x^n + a_{n-1} x^{n-1} + \dots + a_1 x + a_0 = 0.$$

If $r = \frac{c}{d} \in \mathbb{Q}$, $c, d \in \mathbb{Z}$, $d \neq 0$, where c and d have no common divisors, solves this equation,

then c divides a_0 and d divides a_n.

Pf: Since $r = \frac{c}{d}$ solves the equation, we have:

$$a_n \left(\frac{c}{d}\right)^n + a_{n-1}\left(\frac{c}{d}\right)^{n-1} + \dots + a_1 \left(\frac{c}{d}\right) + a_0 = 0.$$

To clear denominators, we multiply both sides by d^n, and obtain:

$$a_n \cdot c^n + a_{n-1} c^{n-1} \cdot d + a_{n-2} \cdot c^{n-2} d^2 + \dots + a_1 c d^{n-1} + a_0 d^n = 0$$

Solving for $a_0 d^n$ to get:

$$a_0 d^n = -\left(a_n \cdot c^n + a_{n-1} c^{n-1} \cdot d + a_{n-2} \cdot c^{n-2} d^2 + \dots + a_1 c d^{n-1}\right)$$

$$= -c \underbrace{\left(a_n c^{n-1} + a_{n-1} c^{n-2} \cdot d + a_{n-2} c^{n-3} d^2 + \dots + a_1 d^{n-1}\right)}_{\in \mathbb{Z}}$$

So it follows that c divides $a_0 d^n$. Since c and d have no common divisors, it follows that c divides a_0.

Solving instead for $a_n c^n$, we find:

$$a_n c^n = -\left(a_{n-1} c^{n-1} \cdot d + a_{n-2} \cdot c^{n-2} d^2 + \cdots + a_1 c d^{n-1} + a_0 d^n\right)$$

$$= -d \underbrace{\left(a_{n-1} c^{n-1} + a_{n-2} c^{n-2} d + \cdots + a_1 c d^{n-2} + a_0 d^{n-1}\right)}_{\in \mathbb{Z}}.$$

This proves that d divides $a_n \cdot c^n$. Since c and d have no common divisors, it follows that c divides a_n. $\square$

<u>Corollary</u>: Let $n \in \mathbb{N}$ and $a_{n-1}, \ldots, a_1, a_0 \in \mathbb{Z}$. Consider the equation

monic $\rightarrow$ $x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0 = 0$

Any rational solution $r \in \mathbb{Q}$ of this equation is an integer that divides a_0.

<u>Pf</u>: By Rational Zeros Theorem, $r = \frac{c}{d} \in \mathbb{Q}$, where $c, d \in \mathbb{Z}$, $d \neq 0$ have no common divisors is such that d divides 1, and c divides a_0. Thus $d = \pm 1$, hence $r = \pm c$. $\square$

Application: Another proof that $\sqrt{2} \notin \mathbb{Q}$.

Consider the equation $x^2 - 2 = 0$.

By the Corollary above, (this is of the form $x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0 = 0$ with $n=2$, and $a_1 = 0$, $a_0 = -2$.) any rational $r \in \mathbb{Q}$ that solves $x^2 - 2 = 0$ must be an integer that divides 2. The only integers that divide 2 are: ± 1, ± 2. Checking all of these, we see that none of them solve $x^2 - 2 = 0$. Therefore, there is no rational solution to $x^2 - 2 = 0$, i.e. $\sqrt{2} \notin \mathbb{Q}$.

Ex: Prove that $\sqrt{17} \notin \mathbb{Q}$. (Using the same approach.)

Sol: Consider the equation $x^2 - 17 = 0$. By the Corollary, any rational solution $r \in \mathbb{Q}$ to this equation is an integer that divides 17; hence one of ± 1, ± 17. Check that none of these solve the equation, and hence $\sqrt{17} \notin \mathbb{Q}$.

Ex: Prove that $\sqrt[3]{6} \notin \mathbb{Q}$. (using the same approach)

Consider the equation $x^3 - 6 = 0$. By the Corollary, any rational solution $r \in \mathbb{Q}$ to this equation must be an integer that divides 6; i.e., one of

± 1, ± 2, ± 3, ± 6. Check that none of these solve the equation, and thus $\sqrt[3]{6} \notin \mathbb{Q}$.