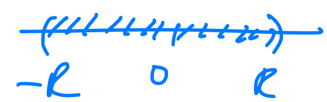
MAT 320/640 Lecture 16 11/1/2021

Uniform Convergence

Last time: Q: We know that $f(x)=\sum_{n=0}^{+\infty} a_n x^n$ is a well-defined function if $|x|<R$, but is it continuous? Differentiable?

$(-R,R)$



More generally, consider a sequence of functions $f_n(x)$.

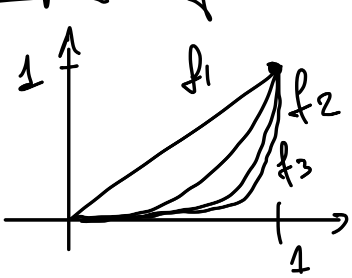
e.g., $f_n(x)=\sum_{k=0}^{n} a_k x^k$ seq. of partial sums of a power series

$f_n(x)=x^n$, $\quad f_n(x)=\sin(nx)$, $\quad f_n(x)=(1-|x|)^n$

Def: The sequence $(f_n(x))_{n\in\mathbb{N}}$ converges pointwise to $f(x)$ on S if $\lim_{n\to\infty} f_n(x)=f(x)$ for all $x\in S$.

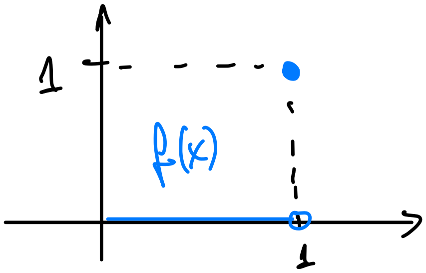
(write $f_n \longrightarrow f$ pointwise)

Example: $f_n(x)=x^n$ $\quad S=[0,1]$



$f_1(x)=x$ $\quad f_3(x)=x^3$

$f_2(x)=x^2$ $\quad f_4(x)=x^4$



$$f(x) = \begin{cases} 0 & \text{if} \quad x \in [0,1) \\ 1 & \text{if} \quad x = 1 \end{cases}$$

(discontinuous at $x=1$)

$f_n \longrightarrow f$ pointwise because:

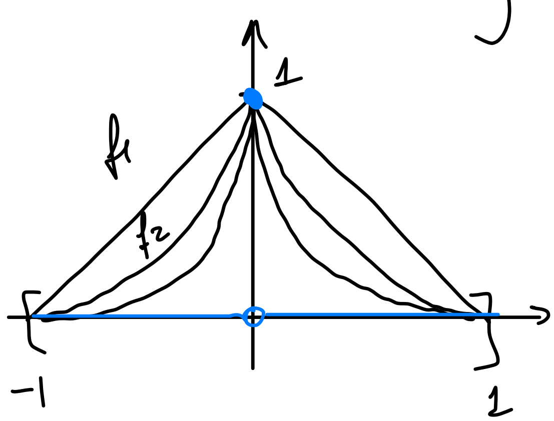
- if $0 \le x < 1$, then $x^n \longrightarrow 0$

while if $x = 1$, then $x^n = 1$.

Rmk: In general, the limit (pointwise) of a sequence of continuous fcts need not be continuous!

Example. $f_n(x) = (1-|x|)^n$ on $S = [-1, 1]$.

Similarly to the above:



$f_1(x) = 1 - |x|$

$f_2(x) = (1-|x|)^2$

$f_3(x) = (1-|x|)^3$

$$f(x) = \begin{cases} 0 & x \neq 0 \\ 1 & x = 0 \end{cases}$$

$f_n \longrightarrow f$ pointwise.

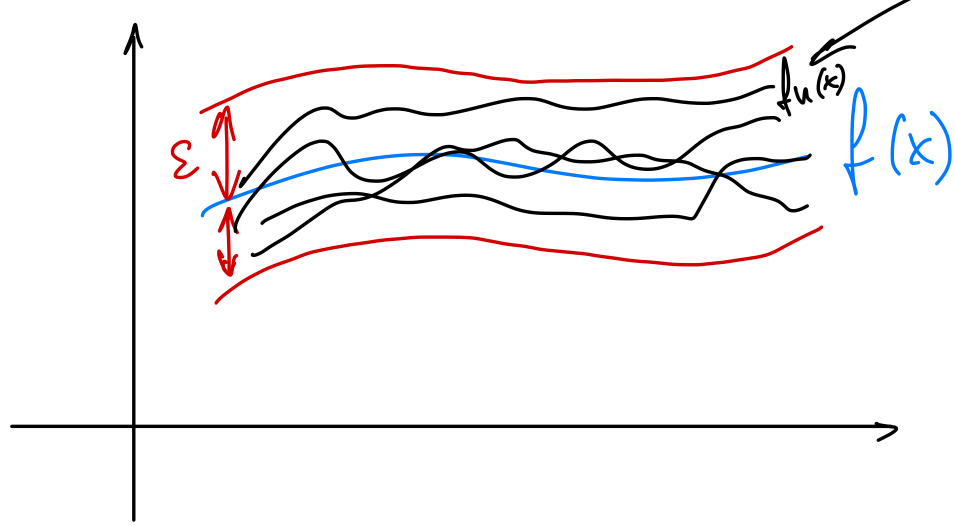
Note: By definition, $f_n(x) = \sum_{k=0}^{n} a_k x^k$ converges pointwise to $f(x) = \sum_{k=0}^{+\infty} a_k x^k = \lim_{n\to\infty} \sum_{k=0}^{n} a_k x^k$ for all $|x| < R$, where $R = \frac{1}{\beta}$, $\beta = \limsup_{n\to\infty} |a_n|^{\frac{1}{n}}$.

Example: $f_n(x) = \sin(nx)$

This example was suggested on the first video as an exercise. Here is the solution:

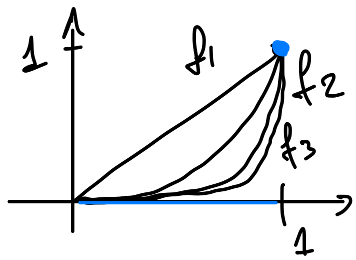
The sequence $(\sin(nx))_{n\in\mathbb{N}}$ converges if and only if x is an integer multiple of π, i.e. if and only if $\sin(nx) \equiv 0$ for all $n \in \mathbb{N}$. Thus, $f_n(x)$ above does not converge (pointwise) to a limit function on any set $S \subset \mathbb{R}$ that is not contained in $\mathbb{Z}\pi = \{m\pi : m \in \mathbb{Z}\}$. If $S \subset \mathbb{Z}\pi$, then $f_n \equiv 0$ so $f_n \to f \equiv 0$ trivially.

Def: The sequence $(f_n(x))_{n\in\mathbb{N}}$ converges uniformly to $f(x)$ on the set S if $\forall \varepsilon > 0$ $\exists N \in \mathbb{N}$ such that if $n \geq N$, then $|f_n(x) - f(x)| < \varepsilon$ for all $x \in S$.



if $n \geq N$, then the graph of $f_n(x)$ is entirely contained in a strip of width ε around the graph of $f(x)$

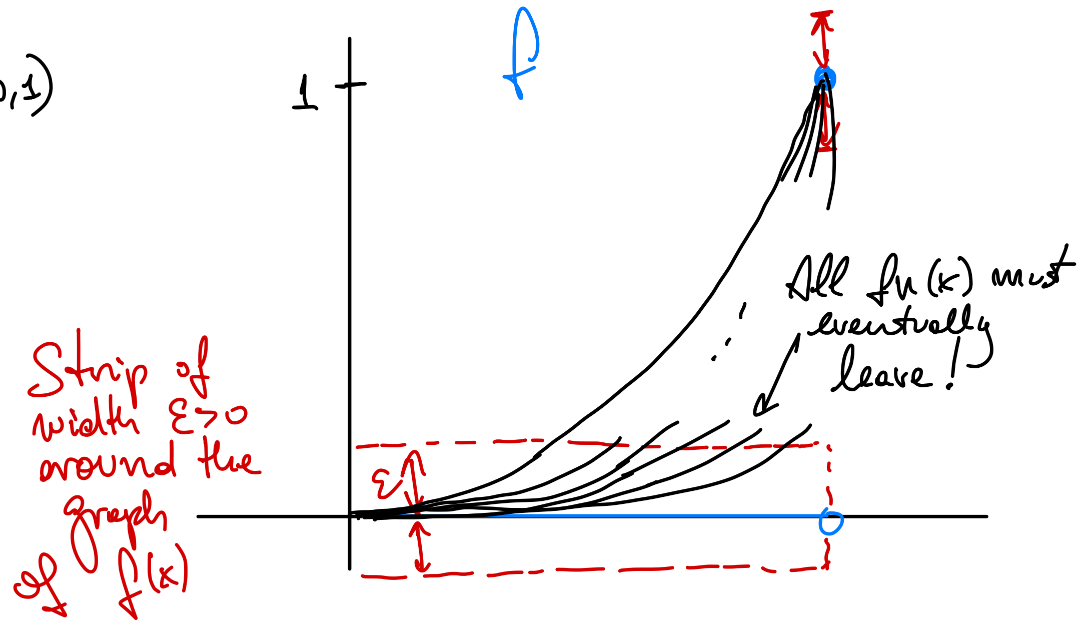
Examples. $f_n(x) = x^n$ on $S = [0,1]$



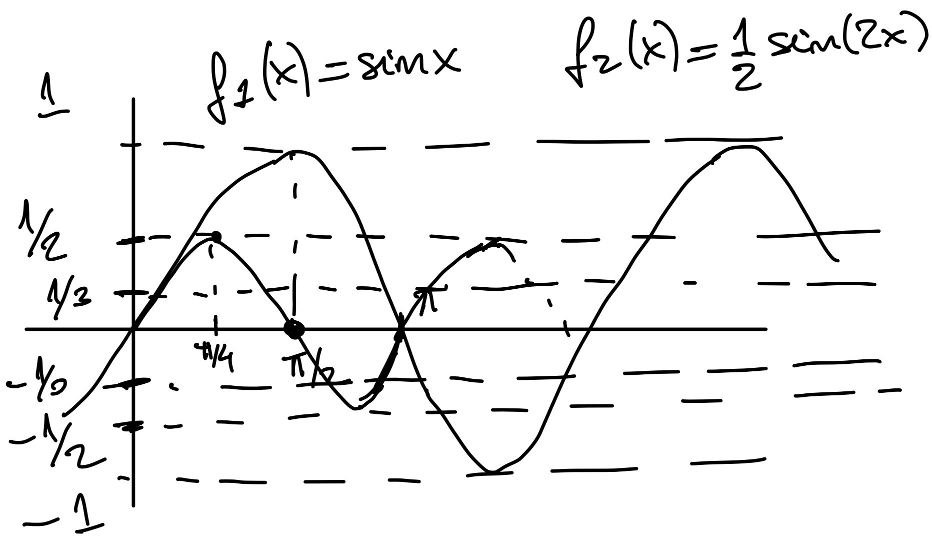
$f_n \to f$ pointwise

$f_n \to f$ does not converge uniformly:

$$f(x) = \begin{cases} 0 & \text{if } x \in [0,1) \\ 1 & \text{if } x = 1 \end{cases}$$



$f_n(x) = \frac{1}{n}\sin(nx)$ converges uniformly to $f(x) \equiv 0$:



$f_n(x)$ oscillates more frequently as $n \nearrow +\infty$, but the amplitude of these oscillations goes to zero (uniformly).

Rigorously: $\quad -\frac{1}{n} \leq f_n(x) \leq \frac{1}{n} \quad \forall x \in \mathbb{R}$

Clearly, $f_n \to f$ pointwise. But also uniformly:
Given $\varepsilon > 0$, take $N = \lceil 1/\varepsilon \rceil + 1$, i.e. the smallest integer larger than $1/\varepsilon$. Then, for all $x \in \mathbb{R}$, we have that if $n \geq N$, $|f_n(x)| = \left|\frac{1}{n}\sin(nx)\right| \leq \frac{1}{n} < \varepsilon$

<u>Rmk:</u> $f_n \to f$ uniformly $\Longrightarrow$ $f_n \to f$ pointwise

$\not\Longleftarrow$

<u>Thm</u>. If $f_n \to f$ uniformly on S and $f_n(x)$ are continuous on S, then $f(x)$ is continuous on S.

<u>Pf</u>: Using the triangle inequality twice:

$$|f(x) - f(x_0)| \leq \underbrace{|f(x) - f_n(x)|} + \underbrace{|f_n(x) - f_n(x_0)|}_{\substack{\text{this is controlled} \\ \text{with continuity} \\ \text{of } f_n(x)}} + \underbrace{|f_n(x_0) - f(x_0)|} \qquad (*)$$

These are controlled with unif. convergence $f_n \to f$.

Given $\varepsilon > 0$ we need to find $\delta > 0$ such that if $|x - x_0| < \delta$, then $|f(x) - f(x_0)| < \varepsilon$.

(i.e. this will prove $f(x)$ is continuous at $x = x_0$.)

Since $f_n \to f$ uniformly $\exists N \in \mathbb{N}$ s.t. if $n \geq N$ and $x \in S$ then $|f_n(x) - f(x)| < \varepsilon/3$; and in particular $|f_N(x_0) - f(x_0)| < \varepsilon/3$. Since $f_N(x)$ is continuous at $x = x_0$, we have $\exists \delta > 0$ s.t. if $|x - x_0| < \delta$, then $|f_N(x) - f_N(x_0)| < \varepsilon/3$. Setting $n = N$ in $(\star)$, we see that if $|x - x_0| < \delta$, then:

$$|f(x) - f(x_0)| \leq \underbrace{|f(x) - f_N(x)|}_{< \frac{\varepsilon}{3}} + \underbrace{|f_N(x) - f_N(x_0)|}_{< \frac{\varepsilon}{3}} + \underbrace{|f_N(x_0) - f(x_0)|}_{< \frac{\varepsilon}{3}}$$

$$< 3 \cdot \frac{\varepsilon}{3} = \varepsilon. \qquad \square$$

So one can say: "uniform convergence preserves continuity".

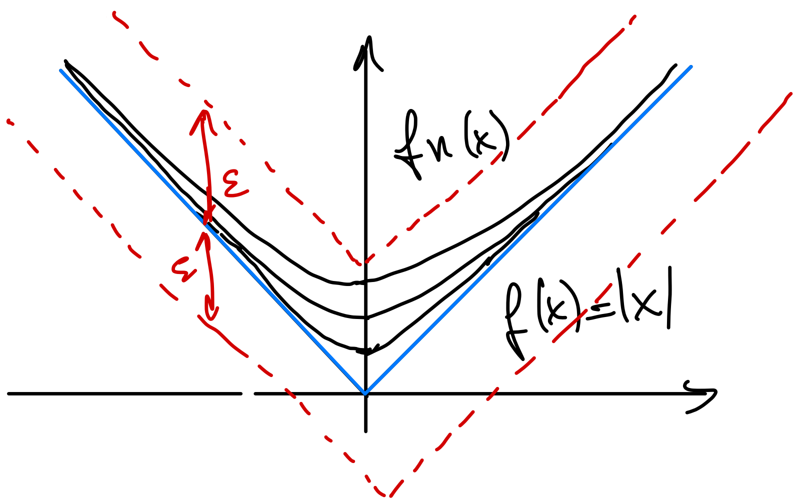
<u>Q</u>: Does <u>uniform convergence</u> preserve differentiability?

<u>A</u>: No.

The seq. $f_n(x) = \sqrt{x^2 + \frac{1}{n}}$ converges pointwise to $f(x) = |x|$.

because:
$$\lim_{n \to \infty} \underbrace{\sqrt{x^2 + \frac{1}{n}}}_{f_n(x)} = \sqrt{x^2} = \underbrace{|x|}_{f(x)}$$

The convergence is uniform:



Even though $f_n(x)$ is differentiable $\forall n \in \mathbb{N}$ and $f_n \to f$ uniformly, $f(x)=|x|$ is <u>not</u> differentiable at $x_0=0$.