MAT320/640 Lecture 15 10/27/2021

Power Series

Def: A <u>power series</u> is a series of the form $\sum_{n=0}^{+\infty} a_n x^n$

$$\sum_{n=0}^{+\infty} a_n x^n$$

Coefficients   variable

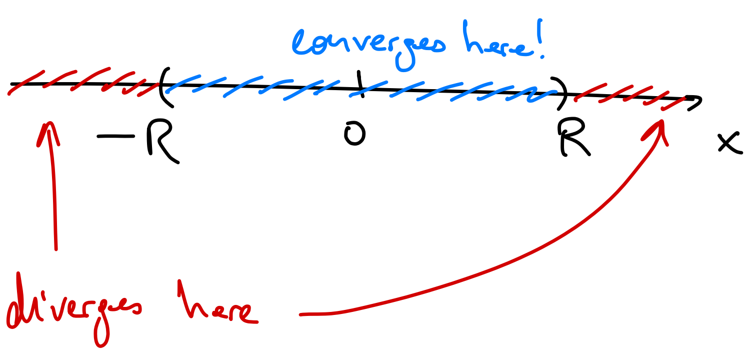
Def (radius of convergence). Consider the power series $\sum_{n=0}^{+\infty} a_n x^n$.

Let $\beta = \limsup_{n\to\infty} |a_n|^{1/n}$, then define $R = \frac{1}{\beta}$.

w/ convention: $\begin{cases} \text{If } \beta = 0, & \text{then } R = \infty \\ \text{If } \beta = \infty, & \text{then } R = 0 \end{cases}$

Proposition. 1) If $|x| < R$, then $\sum_{n=0}^{+\infty} a_n x^n$ converges

2) If $|x| > R$, then $\sum_{n=0}^{+\infty} a_n x^n$ diverges.



Pf: $\sum_{n=0}^{+\infty} a_n x^n$

> Reminder: Root test
>
> $\sum_{n=0}^{+\infty} a_n$ converges absolutely if $\limsup_{n\to\infty} |a_n|^{1/n} < 1$
>
> —— " —— diverges if $\limsup_{n\to\infty} |a_n|^{1/n} > 1$
>
> (test is inconclusive if $\limsup_{n\to\infty} |a_n|^{1/n} = 1$)

$$\limsup_{n\to\infty} |a_n x^n|^{1/n} = \limsup_{n\to\infty} \left(|a_n| \cdot |x^n|\right)^{1/n}$$

$$= \limsup_{n\to\infty} |a_n|^{1/n} \cdot |x|^{\cancel{1/n}\,1}$$

(Def. of radius of convergence)

$$= |x| \cdot \limsup_{n\to\infty} |a_n|^{1/n} = |x| \cdot \frac{1}{R}$$

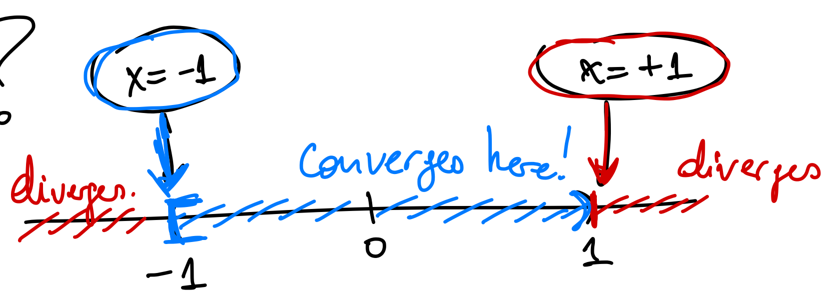
So if $|x| < R$, then $\limsup_{n\to\infty} |a_n x^n|^{1/n} < 1$, hence $\sum_{n=0}^{+\infty} a_n x^n$ converges by Root test.

Similarly, if $|x| > R$, then $\limsup_{n\to\infty} |a_n x^n|^{1/n} > 1$, hence $\sum_{n=0}^{+\infty} a_n x^n$ diverges by the Root test.

$\square$

Q: What if $|x| = R$?

Example: $\sum_{n=1}^{+\infty} \frac{x^n}{n}$



$$a_n = \frac{1}{n} \Rightarrow \beta = \limsup_{n\to\infty} |a_n|^{1/n} = \limsup_{n\to\infty} \left|\frac{1}{n}\right|^{1/n} = \lim_{n\to\infty} \frac{1}{\sqrt[n]{n}} = 1.$$

$$R = \frac{1}{\beta} = 1.$$

p-Series w/ $p = 1$ (Harmonic Series)

If $x = 1$, then $\sum_{n=1}^{+\infty} \frac{x^n}{n} = \sum_{n=1}^{+\infty} \frac{1}{n}$ diverges

p-Series $\sum_{n=1}^{+\infty} \frac{1}{n^p}$ converges if $p > 1$, diverges if $p \leq 1$.

If $x = -1$, then

$$\sum_{n=1}^{+\infty} \frac{x^n}{n} = \sum_{n=1}^{+\infty} \frac{(-1)^n}{n} \quad \text{converges.}$$

"Alternating Harmonic Series"

Q: Can we define $f: [-1, 1) \longrightarrow \mathbb{R}$ as $f(x) = \sum_{n=1}^{+\infty} \frac{x^n}{n}$?

A: Yes.

What is $f(x)$? Is it continuous? Differentiable?

"Heuristics"

$$\sum_{n=0}^{+\infty} x^n = 1 + x + x^2 + x^3 + \cdots \overset{\text{Geom. Series}}{=} \frac{1}{1-x}$$

(if $|x| < 1$)

$$\int \Big(\underbrace{\sum_{n=0}^{+\infty} x^n}_{= \frac{1}{1-x}} \Big) dx \overset{?}{=} \sum_{n=0}^{+\infty} \int x^n \, dx$$

$$= \sum_{n=0}^{+\infty} \frac{x^{n+1}}{n+1} = \sum_{n=1}^{+\infty} \frac{x^n}{n} = f(x)$$

This suggests $f(x) = \int \frac{1}{1-x}\,dx = -\ln(1-x)$

$$= \ln\left(\frac{1}{1-x}\right)$$

Recall: From Lecture 9 (Video 1):

$$(*)\ \liminf_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| \le \liminf_{n\to\infty} |a_n|^{1/n} \le \underbrace{\limsup_{n\to\infty} |a_n|^{1/n}}_{\beta} \le \limsup_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right|$$

Prop: If $\left|\frac{a_{n+1}}{a_n}\right|$ converges, then $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \beta$.

Pf: Recall that $(s_n)_{n\in\mathbb{N}}$ converges if and only if $\liminf_{n\to\infty} s_n = \limsup_{n\to\infty} s_n$. So if $\left|\frac{a_{n+1}}{a_n}\right|$ converges, then $|a_n|^{1/n}$ also converges, and $\lim_{n\to\infty} |a_n|^{1/n} = \beta$. By $(*)$ it follows that $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \beta$. $\square$

Examples: Find the radius of convergence R. Determine if there is convergence at the endpoints $|x| = R$.

a) $\sum_{n=1}^{+\infty} \frac{x^n}{n^2}$ b) $\sum_{n=0}^{+\infty} n!\,x^n$ c) $\sum_{n=0}^{+\infty} \frac{x^{3n}}{2^n}$

$a_n = \frac{1}{n^2}$ $a_n = n!$ $a_n = \begin{cases} 1/2^{n/3} & \text{if } 3 \mid n \\ 0 & \text{if } 3 \nmid n \end{cases}$

$$\sum_{k=0}^{+\infty} \frac{x^{3k}}{2^k} = 1x^0 + 0x^1 + 0x^2 + \frac{x^3}{2^1} + 0x^4 + 0x^5 + \frac{x^6}{2^2} + \cdots$$

$a_0 = 1$ $\quad a_1 = a_2 = 0$ $\quad a_3 = \frac{1}{2}$ $\quad a_4 = a_5 = 0$ $\quad a_6 = \frac{1}{2^2}$

The only powers of x that show up (with a nonzero coeff.) are those of the form $\boxed{n = 3k.}$ ($k = n/3$) Then $a_n = \frac{1}{2^k} = \frac{1}{2^{n/3}}$

If $n = 3k+1$ or $n = 3k+2$, then $a_n = 0$.

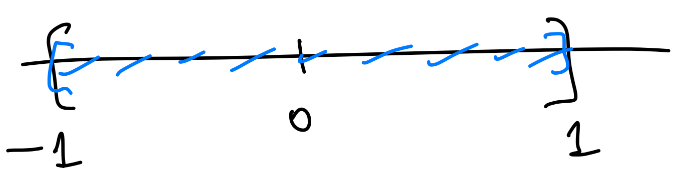
a) $\displaystyle\sum_{n=1}^{+\infty} \frac{x^n}{n^2}$ $\qquad a_n = \frac{1}{n^2}$

$$\beta = \limsup_{n\to\infty} |a_n|^{1/n} = \lim_{n\to\infty} \frac{1}{n^{2/n}} = \Big(\underbrace{\lim_{n\to\infty} \frac{1}{n^{1/n}}}_{1}\Big)^2 = 1.$$

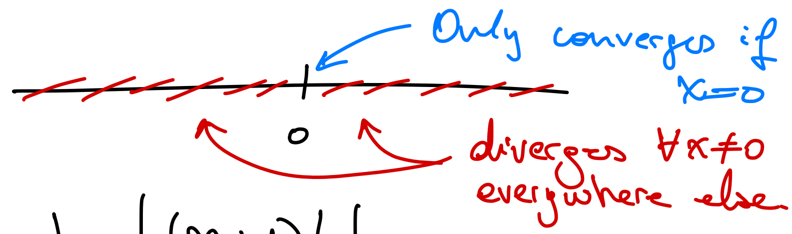
$$R = \frac{1}{\beta} = \frac{1}{1} = 1.$$

$|x| = 1$ $\qquad x = 1: \displaystyle\sum_{n=1}^{+\infty} \frac{1}{n^2}$ Converges.

$x = -1: \displaystyle\sum_{n=1}^{+\infty} \frac{(-1)^n}{n^2}$ Converges.



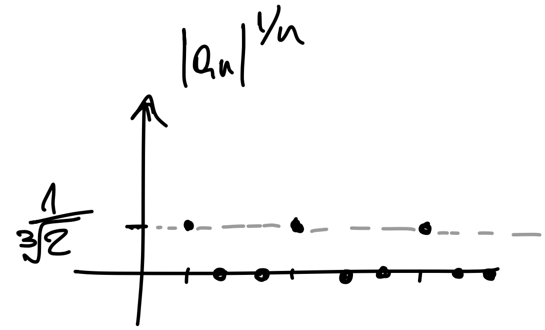
b) $\sum_{n=0}^{+\infty} n!\,x^n$ $\quad a_n = n!$



$\beta = \limsup_{n\to\infty} |a_n|^{1/n}$. $\quad \left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{(n+1)!}{n!}\right| = n+1 \xrightarrow{n\to\infty} \infty$.

So, by the Prop above, $\beta = \infty$;
thus $R = 0$. (No endpoints to analyze).

c) $\sum_{n=0}^{+\infty} \frac{x^{3n}}{2^n}$ $\quad a_n = \begin{cases} 1/2^{n/3} & \text{if } 3\mid n \\ 0 & \text{if } 3\nmid n \end{cases}$



Note: $|a_n|^{1/n}$ does not converge

$\liminf_{n\to\infty} |a_n|^{1/n} = 0$. b/c $a_n = 0$ for infinitely many values of n.

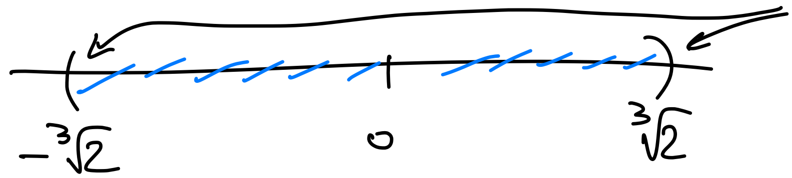
$$\beta = \limsup_{n\to\infty} |a_n|^{1/n} = \lim_{n\to\infty} \left|\frac{1}{2^{n/3}}\right|^{1/n} = \lim_{n\to\infty} \frac{1}{2^{\frac{\cancel{n}}{3}\cdot\frac{1}{\cancel{n}}}} = \frac{1}{\sqrt[3]{2}}$$

these are the only nonzero values of $|a_n|^{1/n}$.

$$\boxed{\text{The radius of conv. is } R = \sqrt[3]{2}}$$

$$R = \frac{1}{\beta} = \frac{1}{1/\sqrt[3]{2}} = \sqrt[3]{2}$$

Endpoints:



diverges.

$x = \sqrt[3]{2}$

$$\sum_{n=0}^{+\infty} \frac{x^{3n}}{2^n} = \sum_{n=0}^{+\infty} \frac{2^{\frac{\cancel{3}n}{\cancel{3}}}}{2^n} = \sum_{n=0}^{+\infty} 1 \quad \text{diverges.}$$

$x = -\sqrt[3]{2}$:

even if $2 | n$
odd if $2 \nmid n$

$$\sum_{n=0}^{+\infty} \frac{\left(-2^{\frac{1}{3}}\right)^{3n}}{2^n} = \sum_{n=0}^{+\infty} \frac{(-1)^{3n} \cdot \cancel{2^{\frac{3n}{3}}}}{\cancel{2^n}} = \sum_{n=0}^{+\infty} (-1)^{3 \cdot n} = \sum_{n=0}^{+\infty} (-1)^n$$

$(-1)^{3n} = \left((-1)^3\right)^n = (-1)^n$

also diverges.

$(1 - 1 + 1 - 1 + 1 - \ldots$ diverges$)$