MAT 320/640 Lecture 14 10/25/2021

Uniform Continuity

Recall: $f(x)$ is continuous at x_0 if $\forall \varepsilon > 0$, $\exists \delta > 0$,

$$|x - x_0| < \delta \implies |f(x) - f(x_0)| < \varepsilon.$$

What does δ depend on? Only on $\varepsilon > 0$, or also on x_0?

Examples: $f(x) = 3x + 1$

$$|f(x) - f(x_0)| = |3x + 1 - (3x_0 + 1)|$$
$$= 3|x - x_0|$$

Set $\delta = \varepsilon/3$, to have that

$$|x - x_0| < \delta \implies |f(x) - f(x_0)| < \varepsilon.$$

Does not depend on x_0, just depends on $\varepsilon > 0$!

$f(x) = \dfrac{1}{x^2}$, $x \in (0, +\infty)$

$$|f(x) - f(x_0)| = \left| \frac{1}{x^2} - \frac{1}{x_0^2} \right|$$
$$= \left| \frac{x_0^2 - x^2}{x^2 x_0^2} \right|$$
$$= \left| \frac{(x_0 - x)(x_0 + x)}{x^2 x_0^2} \right|$$

$$|x - x_0| < \frac{x_0}{2} \overset{\text{Tr. ineq.}}{\implies} \frac{x_0}{2} < |x| < \frac{3x_0}{2}$$
$$\implies |x + x_0| < \frac{5x_0}{2}$$

So:

$$\frac{|x_0 - x| \cdot |x_0 + x|}{x^2 x_0^2} < \frac{|x_0 - x| \cdot \frac{5x_0}{2}}{\left(\frac{x_0}{2}\right)^2 x_0^2}$$
$$= \frac{10|x - x_0|}{x_0^3}$$

This δ depends on x_0 as well as $\varepsilon > 0$.

Thus, we may set

$$\delta = \min\left\{ \frac{x_0^3 \cdot \varepsilon}{10}, \frac{x_0}{2} \right\}$$

then $|x-x_0|<\delta \Longrightarrow |f(x)-f(x_0)|<\varepsilon$.

Definition. The function $f: D\subset\mathbb{R}\to\mathbb{R}$ is uniformly continuous on D if $\forall\varepsilon>0\ \exists\delta>0$ s.t. if $x,y\in D$ with $|x-y|<\delta$, then $|f(x)-f(y)|<\varepsilon$.

Important differences:

	Continuity	Uniform continuity
δ	$\delta=\delta(\varepsilon,x_0)$	$\delta=\delta(\varepsilon)$ depends only on ε
where?	At each point $x_0\in D$	Only on subsets D, not individual points!

Examples: $f(x)=3x+1$ is uniformly continuous on $\mathbb{R}$

$f(x)=\dfrac{1}{x^2}$ is uniformly continuous on domains of the form $[a,+\infty)$, $a>0$.

$f(x)$ is uniformly cont. here.

If $x_0 \geq a$, then we see:

$$\min\left\{\frac{x_0^3 \cdot \varepsilon}{10}, \frac{x_0}{2}\right\} \geq \underbrace{\min\left\{\frac{a^3\varepsilon}{10}, \frac{a}{2}\right\}}.$$

Set δ to be this! Then δ depends only on ε. (and a)

If $x, y \in [a, +\infty)$ satisfy

$$|x-y| < \delta = \min\left\{\frac{a^3\varepsilon}{10}, \frac{a}{2}\right\}, \text{ then } |f(x)-f(y)| < \varepsilon.$$

__Theorem.__ If $f(x)$ is __continuous__ on a closed interval $[a,b]$, then $f(x)$ is __uniformly continuous__ on $[a,b]$.

__Pf:__ Suppose, by contradiction, that $f(x)$ is not unif. cont. on $[a,b]$.

NOT$\left(\forall \varepsilon>0 \ \exists \delta>0 \ x,y\in[a,b],\ |x-y|<\delta \Rightarrow |f(x)-f(y)|<\varepsilon\right)$

$=\left(\exists \varepsilon>0 \ \forall \delta>0 \ \exists x,y\in[a,b] \ \ |x-y|<\delta \ \underline{\text{and}} \ |f(x)-f(y)|\geq\varepsilon\right)$

Then, $\exists \varepsilon>0$ s.t. $\forall \delta>0$, $\exists x,y\in[a,b]$ with $|x-y|<\delta$ and $|f(x)-f(y)|\geq\varepsilon$. Taking $\delta=\frac{1}{n}$ for every $n\in\mathbb{N}$, we get sequences $x_n, y_n \in [a,b]$ with $|x_n - y_n| < \frac{1}{n}$ and $|f(x_n)-f(y_n)| \geq \varepsilon$. By the Bolzano-Weierstrass Thm, since (x_n) is __bounded__, it has a convergent subsequence x_{n_k}, say $x_{n_k} \longrightarrow x_0$. Then, since $|x_{n_k} - y_{n_k}| < \frac{1}{n_k}$

we also have $y_{n_k} \to x_0$. By continuity of $f(x)$,

$$x_{n_k} \to x_0 \implies f(x_{n_k}) \to f(x_0)$$

$$y_{n_k} \to x_0 \implies f(y_{n_k}) \to f(x_0)$$

so $\lim\limits_{n_k \to \infty} |f(x_{n_k}) - f(y_{n_k})| = 0$. But $|f(x_{n_k}) - f(y_{n_k})| \geq \varepsilon$, which gives the desired contradiction. $\square$

Example: $f(x) = \frac{1}{x^2}$ is unif. cont. on $[1, 100]$.

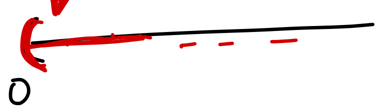
$g(x) = x^{2021} \sin(e^{x+7}) - x^6 + 1$ is unif. cont. on $[a,b]$.

Theorem. If $f(x)$ is uniformly cont. on D and $(s_n)_{n\in\mathbb{N}}$ is a Cauchy sequence in D, then $(f(s_n))_{n\in\mathbb{N}}$ is a Cauchy sequence.

"Unif. cont. functions take Cauchy seq. to Cauchy seq."

Pf. If $(s_n)_{n\in\mathbb{N}}$ is Cauchy, then $\forall \delta > 0$ $\exists N \in \mathbb{N}$ s.t. $n, m \geq N \implies |s_n - s_m| < \delta$. Since $f(x)$ is unif. cont., $\forall \varepsilon > 0$ $\exists \delta > 0$ s.t. $|x-y| < \delta \implies |f(x) - f(y)| < \varepsilon$. Applying this δ to the Cauchy seq. def. above, we find $N \in \mathbb{N}$ s.t. $n, m \geq N$ imply $|s_n - s_m| < \delta$, So $|f(s_n) - f(s_m)| < \varepsilon$. In other words, the sequence $(f(s_n))_{n\in\mathbb{N}}$ is Cauchy. $\square$

Problem is near $x=0$.

<u>Example</u>: $f(x) = \frac{1}{x^2}$ on 

or $(0,1)$

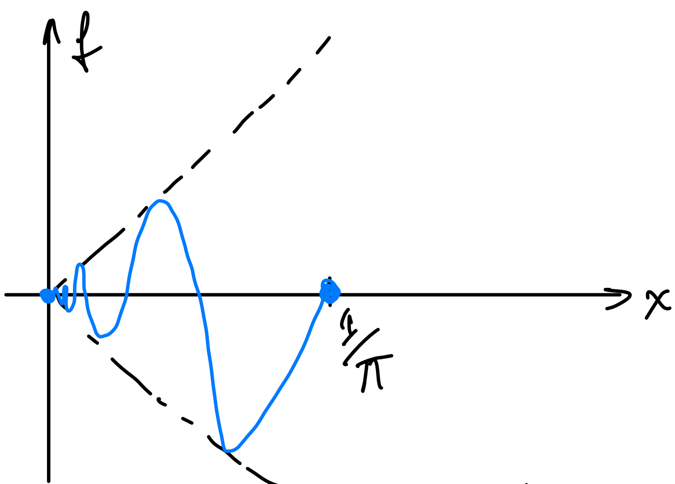
<u>Claim</u>: $f(x)$ is <u>not</u> unif. cont. on $(0,+\infty)$.

Take $s_n = \frac{1}{n}$. This is a Cauchy sequence because it is convergent $(s_n \to 0)$. However, $f(s_n) = \frac{1}{s_n^2} = n^2$ and the sequence n^2 is not Cauchy. $\square$

Extending functions from (a,b) to $[a,b]$.

$f : (0,\frac{1}{\pi}) \to \mathbb{R}$

$f(x) = x\sin\left(\frac{1}{x}\right)$

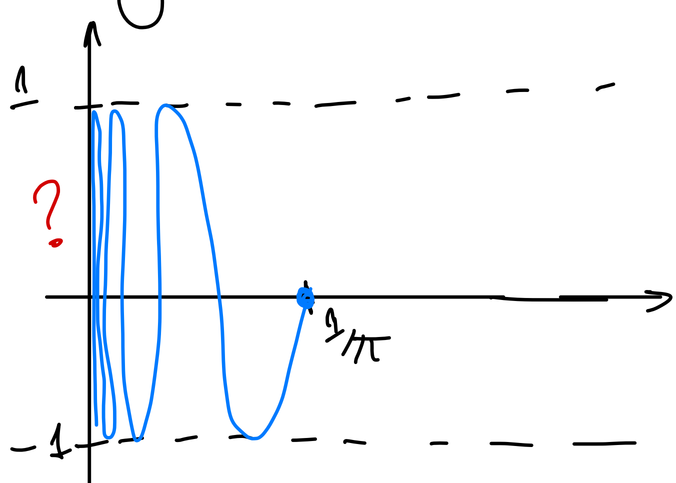


$\tilde{f} : [0,\frac{1}{\pi}] \to \mathbb{R}$, $\tilde{f}(x) = f(x)$ $\forall x \in (0,\frac{1}{\pi})$

$\tilde{f}(0) = 0$, $\tilde{f}\left(\frac{1}{\pi}\right) = 0$.

$g : (0,\frac{1}{\pi}) \to \mathbb{R}$

$g(x) = \sin\left(\frac{1}{x}\right)$



We can extend g at $x = 1/\pi$ but <u>not</u> at $x=0$ keeping it <u>continuous</u>.

Theorem: A continuous function $f:(a,b)\to\mathbb{R}$ can be extended to a continuous function $\tilde{f}:[a,b]\to\mathbb{R}$ if and only if f is uniformly continuous on (a,b).

Proof: If f can be extended to a cont. $\tilde{f}:[a,b]\to\mathbb{R}$, then f is unif. cont. because $\tilde{f}:[a,b]\to\mathbb{R}$ is a continuous function on a closed interval.

Conversely, suppose $f:(a,b)\to\mathbb{R}$ is unif. cont.

Claim 1. If $(s_n)_{n\in\mathbb{N}}$ is a sequence on (a,b) that converges to a, then $(f(s_n))_{n\in\mathbb{N}}$ converges.

Indeed, if $s_n\to a$, then s_n is Cauchy. As f is unif. cont., it follows that $f(s_n)$ is Cauchy, so $f(s_n)$ is convergent.

Claim 2. If $(s_n)_{n\in\mathbb{N}}$ and $(t_n)_{n\in\mathbb{N}}$ are sequences in (a,b) converging to a, then $\lim_{n\to\infty} f(s_n)=\lim_{n\to\infty} f(t_n)$.

By Claim 1, $f(s_n)\to y_0$ and $f(t_n)\to y_1$.

Construct a new sequence $(u_n)_{n\in\mathbb{N}}$ by intertwining s_n and t_n:

$$(u_n)_{n\in\mathbb{N}}=(s_1,t_1,s_2,t_2,s_3,t_3,\dots,s_n,t_n,\dots)$$

Clearly, $u_n \longrightarrow a$. By Claim 1, $f(u_n)$ is Cauchy and hence converges. Now, since $f(u_n)$ is the sequence $(f(u_n))_{n\in\mathbb{N}} = (f(s_1), f(t_1), f(s_2), f(t_2), \dots)$, and $f(s_n) \longrightarrow y_0$, $f(t_n) \longrightarrow y_1$, it follows that $y_0 = y_1$, as desired.

To extend f to $x=a$, we define:

$$\tilde{f}(a) = \lim_{n\to\infty} f(s_n)$$

where $(s_n)_{n\in\mathbb{N}}$ is any sequence s.t. $s_n \longrightarrow a$.

By Claim 1, the limit exists.

By Claim 2, the limit does not depend on the choice of the sequence $(s_n)_{n\in\mathbb{N}}$.

Proceed analogously to extend f to $x=b$. $\square$