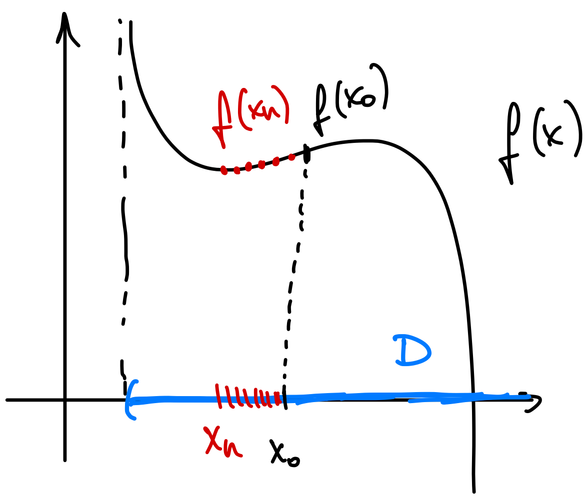
MAT 320/640 Lecture 12 10/18/2021

Continuous Functions

Def: A function $f: D \subset \mathbb{R} \to \mathbb{R}$ is <u>continuous</u> at $x_0 \in D$ if for every sequence $(x_n)_{n\in\mathbb{N}}$ in D such that $x_n \to x_0$, we have $f(x_n) \to f(x_0)$. If $f(x)$ is continuous at every $x_0 \in D$, then we say $f(x)$ is continuous (on D).



Both definitions of continuity (w/ sequences, above, and w/ $\varepsilon - \delta$ are equivalent).

Theorem: A function $f: D \subset \mathbb{R} \to \mathbb{R}$ is continuous at $x_0 \in D$ if and only if $\forall \varepsilon > 0$, $\exists \delta > 0$ such that if $x \in D$ satisfies $|x - x_0| < \delta$, then $|f(x) - f(x_0)| < \varepsilon$.

Pf: $(\forall \varepsilon > 0\ \exists \delta > 0 \ldots) \Longrightarrow (f \text{ is cont.})$

Given $\varepsilon > 0$, there exists $\delta > 0$ such that if $x \in D$, $|x - x_0| < \delta$, then $|f(x) - f(x_0)| < \varepsilon$. If $x_n \in D$ is a sequence such that $x_n \to x_0$, then there exists

$N \in \mathbb{N}$ such that $n > N \implies |x_n - x_0| < \delta$. This implies $|f(x_n) - f(x_0)| < \varepsilon$, i.e. $f(x_n) \to f(x_0)$.

Conversely, we now need to prove that

$$(\forall \varepsilon > 0 \ \exists \delta > 0 \ldots) \Longleftarrow (f \text{ is cont.})$$

Suppose, by contradiction, that f is cont. at x_0, but $(\forall \varepsilon > 0 \ \exists \delta > 0 \ldots)$ fails, i.e. $\exists \varepsilon > 0$ s.t. $\forall \delta > 0$, if $x \in D$, $|x - x_0| < \delta$ but $|f(x) - f(x_0)| \geq \varepsilon$. Take $\delta = \frac{1}{n}$ for each $n \in \mathbb{N}$, and let $x_n \in D$ be the corresponding $x \in D$ with $|x_n - x_0| < \frac{1}{n}$ and $|f(x_n) - f(x_0)| \geq \varepsilon > 0$. In other words, $x_n \to x_0$ but $f(x_n) \not\to f(x_0)$, giving the desired contradiction to f being continuous.

$\square$

<u>Example:</u> Prove that $f(x) = 2x^2 + 1$ is continuous

(i) with definition (using sequences),

(ii) with $\varepsilon - \delta$.

(i) Given $x_n \to x_0$, note that $f(x_n) = 2x_n^2 + 1$, so

$$\lim_{n \to \infty} \underbrace{2x_n^2 + 1}_{f(x_n)} = 2 \cdot \Big(\underbrace{\lim_{n \to \infty} x_n}_{x_0}\Big)^2 + 1 = 2x_0^2 + 1 = f(x_0);$$

Properties of limits of sequences from past lectures

i.e. $f(x)$ is continuous.

(ii) Given $\varepsilon>0$, we need to find $\delta>0$ such that
$$|x-x_0|<\delta \implies |f(x)-f(x_0)|<\varepsilon.$$

$$\left.\begin{aligned} f(x)&=2x^2+1\\ f(x_0)&=2x_0^2+1\end{aligned}\right\} \implies |f(x)-f(x_0)| = |2x^2+1-(2x_0^2+1)|$$
$$= |2x^2-2x_0^2|$$

Need to bound independently of x:
$$= 2|x^2-x_0^2|$$
$$= 2\underbrace{|x-x_0|}_{<\delta}\cdot\underbrace{|x+x_0|}_{??}$$

$$|x+x_0| \le 1+2|x_0|.$$

$$\le 2|x-x_0|\cdot(1+2|x_0|)<\varepsilon$$

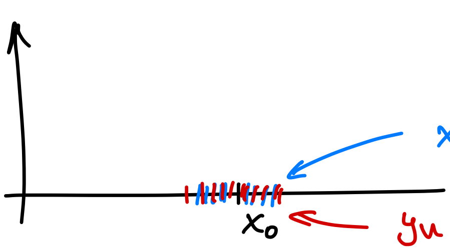
(if $|x-x_0|<1$, then
$$|x| = |x-x_0+x_0| \le \underbrace{|x-x_0|}_{<1}+|x_0| < \underbrace{1+|x_0|}_{\text{no dependence on } x!}$$
so $|x+x_0| \le |x|+|x_0| < 1+2|x_0|$.)

Solve for $|x-x_0|$:
$$|x-x_0| < \boxed{\frac{\varepsilon}{2(1+2|x_0|)}}$$
$$\delta \le \frac{\varepsilon}{2(1+2|x_0|)}$$

Given $\varepsilon>0$, let $\delta = \min\left\{1, \dfrac{\varepsilon}{2(1+2|x_0|)}\right\}$ and it follows that if $|x-x_0|<\delta$, then $|f(x)-f(x_0)|<\varepsilon$. $\square$

<u>Examples of functions that are not continuous:</u>

$$f(x)=\begin{cases} 1 & \text{if } x\in\mathbb{Q}\\ 0 & \text{if } x\notin\mathbb{Q}\end{cases}$$
is <u>not</u> continuous at any $x_0\in\mathbb{R}$, since there exist sequences $x_n\in\mathbb{Q}$, $y_n\notin\mathbb{Q}$ with $x_n\to x_0$, $y_n\to x_0$.



$x_n \to x_0$ such that $x_n\in\mathbb{Q}$, $\forall n\in\mathbb{N}$

$y_n \to x_0$ such that $y_n\notin\mathbb{Q}$, $\forall n\in\mathbb{N}$

Note $f(x_n)=1$ and $f(y_n)=0$, so no matter if $x_0 \in \mathbb{Q}$ or $x_0 \notin \mathbb{Q}$, at least one of them is going to violate the condition $f(x_n) \to f(x_0)$ or $f(y_n) \to f(x_0)$.

$$f(x)=\begin{cases} \frac{1}{x}\sin\left(\frac{1}{x^2}\right) & \text{if } x \neq 0 \\ 0 & \text{if } x=0 \end{cases}$$

is continuous at all $x_0 \neq 0$, but it is <u>not</u> continuous at $x_0=0$.

It suffices to show a sequence x_n such that $x_n \to x_0 = 0$ but $f(x_n) \not\to f(x_0)=0$. Let us find x_n such that $f(x_n)=\frac{1}{x_n}$; namely we solve:

$$\frac{1}{\cancel{x_n}}\sin\left(\frac{1}{x_n^2}\right)=\frac{1}{\cancel{x_n}} \iff \frac{1}{x_n^2}=2\pi n+\frac{\pi}{2}$$

Let $x_n = \dfrac{1}{\sqrt{\frac{\pi}{2}+2\pi n}}$ for all $n \in \mathbb{N}$.

$$\lim_{n\to\infty} x_n = \lim_{n\to\infty}\frac{1}{\sqrt{\frac{\pi}{2}+\underbrace{2\pi n}_{\to\infty}}}=0.$$

So, altogether, we have $x_n \to 0$ but $f(x_n)=\frac{1}{x_n}$

So $\lim\limits_{n\to\infty} f(x_n)=\lim\limits_{n\to\infty}\frac{1}{x_n}=+\infty$. $(\neq f(x_0)=0)$.

<u>Exercise</u>: How about

$$f(x)=\begin{cases} x^p\sin\left(\frac{1}{x}\right) & \text{if } x\neq 0 \\ 0 & \text{if } x=0 \end{cases}$$

For what values of p is it continuous at $x_0=0$?

Theorem. If $f : D \subset \mathbb{R} \to \mathbb{R}$ is continuous at $x_0 \in D$, then so are $|f|$ and $c \cdot f$, $c \in \mathbb{R}$.

Pf: If $x_n \to x_0$, then $|f(x_n)| \longrightarrow |f(x_0)|$ because:

$$\lim_{n\to\infty} |f(x_n)| = \Big| \underbrace{\lim_{n\to\infty} f(x_n)}_{f(x_0)} \Big| = |f(x_0)|.$$

$$\big||f(x_n)| - |f(x_0)|\big| \leq |f(x_n) - f(x_0)|, \forall n$$

Moreover, $\displaystyle\lim_{n\to\infty} c \cdot f(x_n) = c \cdot \underbrace{\lim_{n\to\infty} f(x_n)}_{f(x_0)} = c \cdot f(x_0)$.

$\square$

Theorem. If f and g are continuous at $x_0 \in \mathbb{R}$, so are $f+g$, $f \cdot g$, and $\dfrac{f}{g}$ (provided $g(x_0) \neq 0$).

Pf: Let $x_n \to x_0$, and note that:

$$\begin{aligned}
\lim_{n\to\infty} (f+g)(x_n) &= \lim_{n\to\infty} f(x_n) + g(x_n) \\
&= \lim_{n\to\infty} f(x_n) + \lim_{n\to\infty} g(x_n) \\
&= f(x_0) + g(x_0) \\
&= (f+g)(x_0).
\end{aligned}$$

Similarly, the continuity of $f \cdot g$ and $\dfrac{f}{g}$ follows from properties of limits of products and ratios.

$\square$

Theorem. If $f(x)$ is continuous at x_0,
and $g(x)$ is continuous at $f(x_0)$,
then $(g\circ f)(x)$ is continuous at x_0.

Pf: Let $x_n \to x_0$. By continuity of $f(x)$ at x_0, we have $f(x_n) \to f(x_0)$. Let $y_n = f(x_n)$ and note $y_n \to f(x_0)$. So by continuity of $g(x)$ at $f(x_0)$, we have that $\underbrace{g(y_n)}_{= g(f(x_n))} \to g(f(x_0))$.

Therefore, $(g\circ f)(x_n) \to (g\circ f)(x_0)$;
that is, $(g\circ f)$ is continuous at x_0. $\square$

Theorem. If f and g are continuous at $x_0 \in \mathbb{R}$, so are the functions $\max\{f,g\}$ and $\min\{f,g\}$.

Pf: (Trick: write down an analytic formula for max/min)

$$\max\{a,b\} = \frac{1}{2}(a+b) + \frac{1}{2}|a-b|$$

$$\min\{a,b\} = \frac{1}{2}(a+b) - \frac{1}{2}|a-b|$$

$$|a-b| = \begin{cases} a-b, & \text{if } a \geq b \\ b-a, & \text{if } a<b \end{cases}$$

$$\frac{1}{2}(a+b) + \frac{1}{2}|a-b| = \begin{cases} \frac{1}{2}(a+\cancel{b}) + \frac{1}{2}(a-\cancel{b}), & \text{if } a \geq b \\ \frac{1}{2}(\cancel{a}+b) + \frac{1}{2}(b-\cancel{a}), & \text{if } a<b \end{cases}$$

$$= \begin{cases} a & \text{if } a \geq b \\ b & \text{if } a<b \end{cases} = \max\{a,b\}.$$

So $\max\{f,g\} = \frac{1}{2}(f+g) + \frac{1}{2}|f-g|$ is a composition (sum, difference, product with constant, absolute value of, and composition) of continuous functions, hence is itself continuous by the results proved above. Similarly for $\min\{f, g\}$.

$\square$