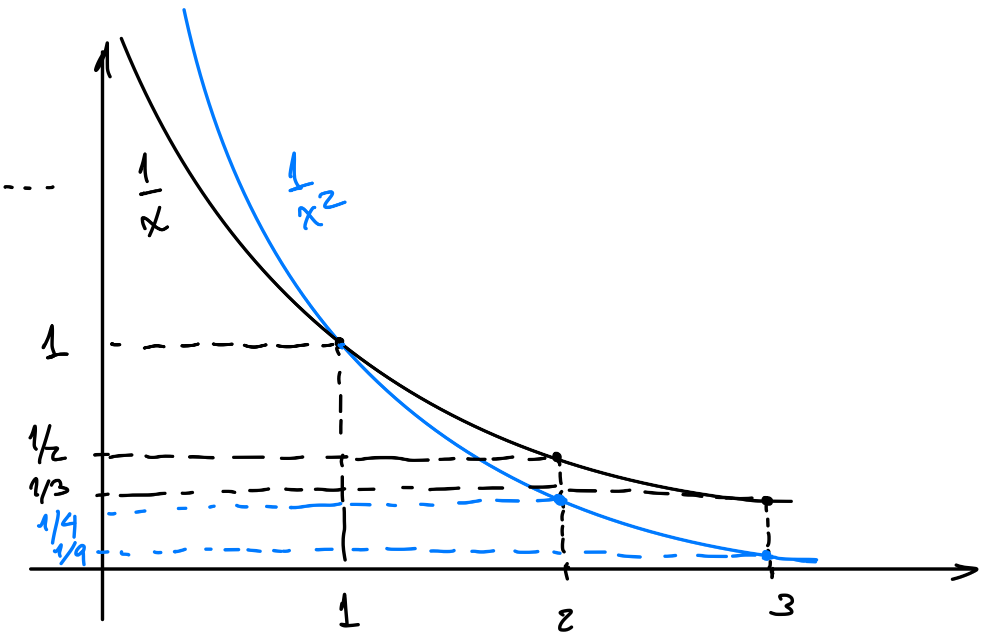
MAT320/640 # Lecture 11 10/13/2021

p - Series $\quad p > 0$

$$\sum_{n=1}^{\infty} \frac{1}{n^p} = 1 + \frac{1}{2^p} + \frac{1}{3^p} + \frac{1}{4^p} + \cdots$$

$$\boxed{\begin{array}{c} a_n = \frac{1}{n^p} = f(n) \\ f(x) = \frac{1}{x^p} = x^{-p} \end{array}}$$

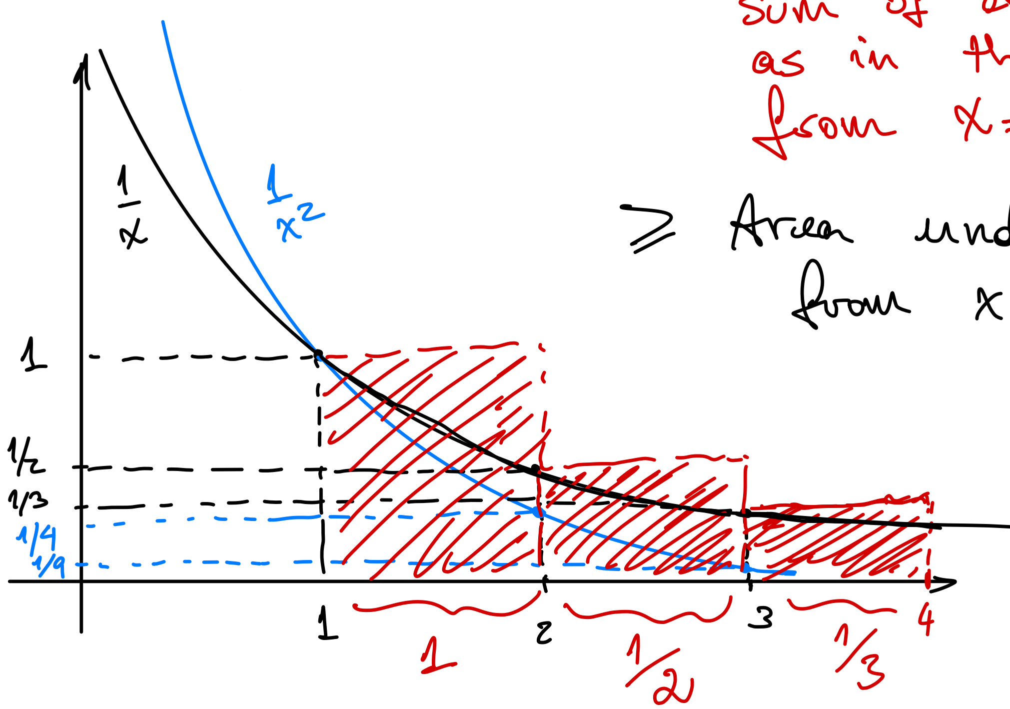


p = 1: "Harmonic Series"

$$\sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots = \lim_{n \to \infty} S_n$$

$$S_n = \sum_{k=1}^{n} \frac{1}{k} = \underbrace{1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}}$$

sum of areas of red rectangles as in the picture below from $x = 1$ up to $x = n+1$



$\geq$ Area under $f(x) = \frac{1}{x}$ from $x = 1$ up to $x = n+1$

$$= \int_1^{n+1} \frac{1}{x}\, dx$$

$$= \ln(n+1)$$

Therefore, $s_n \geq \ln(n+1)$ for all $n \in \mathbb{N}$. Taking limits as $n \nearrow +\infty$ on both sides, we find:

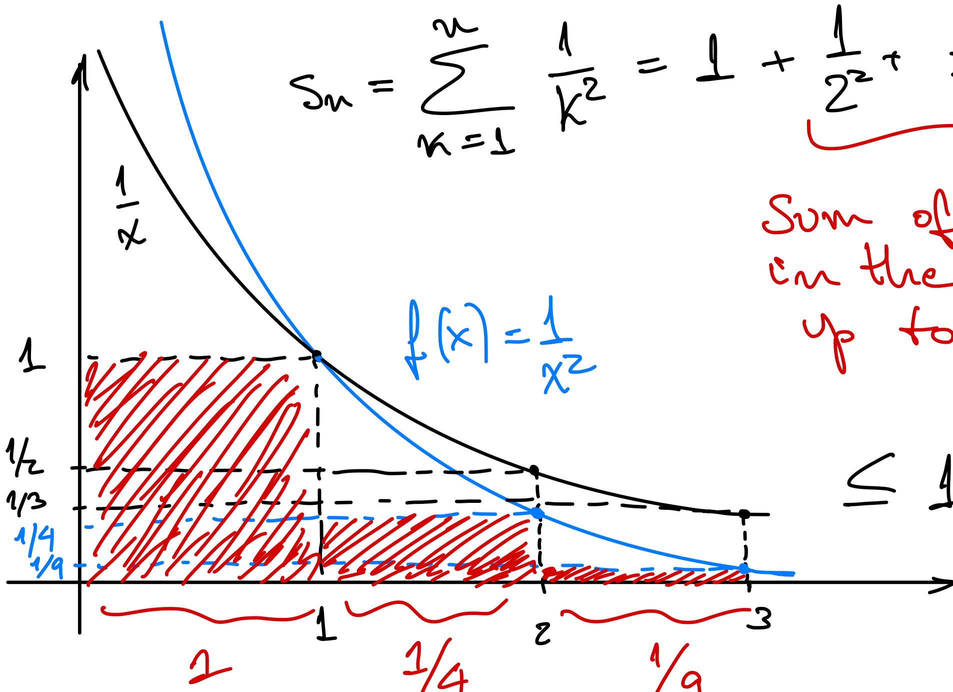
$$\lim_{n\to\infty} s_n \geq \lim_{n\to\infty} \ln(n+1) = +\infty$$

Thus $\lim_{n\to\infty} s_n = +\infty$, and hence $\sum_{n=1}^{+\infty} \frac{1}{n} = +\infty$.

<u>$p=2$</u> : $$\sum_{n=1}^{+\infty} \frac{1}{n^2} = 1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \cdots$$

$$s_n = \sum_{k=1}^{n} \frac{1}{k^2} = 1 + \underbrace{\frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \cdots + \frac{1}{n^2}}$$

Sum of areas of red rectangles in the picture, from $x=1$ up to $x=n$.



$$\leq 1 + \text{Area under } f(x) \text{ from } x=1 \text{ to } x=n$$

$$= 1 + \int_1^n \frac{1}{x^2}\,dx$$

$$\int x^{-2}\,dx = \frac{x^{-2+1}}{-2+1} = -\frac{1}{x}$$

$$= 1 + \left(-\frac{1}{x}\right)\Big|_1^n$$

$$= 1 - \frac{1}{n} + 1 = 2 - \frac{1}{n}.$$

Therefore, $s_n \leq 2 - \frac{1}{n}$ for all $n \in \mathbb{N}$, so taking limits as $n \nearrow +\infty$, we find:

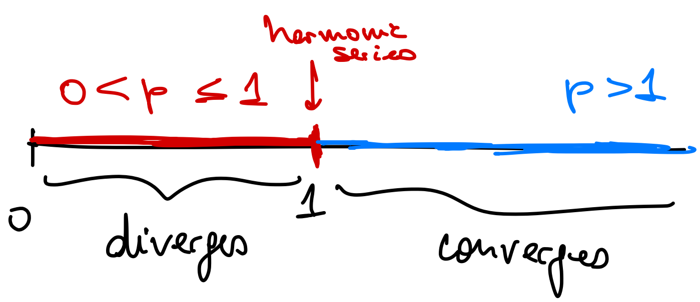
$$\lim_{n\to\infty} s_n \leq \lim_{n\to\infty} 2 - \frac{1}{n} = 2$$

Thus $(s_n)_{n\in\mathbb{N}}$ is an increasing sequence, bounded from above by 2, therefore it converges.

(It can be shown that $\lim_{n\to\infty} s_n = \frac{\pi^2}{6} < 2$).

<u>Theorem</u>. The p-series $\sum_{n=1}^{+\infty} \frac{1}{n^p}$ converges if and only if $p>1$.

<u>Proof</u>:



If $p>1$, then

$$s_n = \sum_{k=1}^{n} \frac{1}{k^p} = 1 + \frac{1}{2^p} + \frac{1}{3^p} + \cdots + \frac{1}{n^p}$$

$$\leq 1 + \int_1^n \frac{1}{x^p}\,dx = 1 + \left(\frac{x^{1-p}}{1-p}\right)\Bigg|_1^n =$$

$$= 1 + \frac{n^{\overset{<0}{1-p}}}{1-p} - \frac{1}{1-p} \xrightarrow{n \nearrow +\infty} 1 + \frac{1}{p-1}$$

The sequence of partial sums $(s_n)_{n\in\mathbb{N}}$ is increasing and bounded from above by $1+\frac{1}{p-1}$ and thus converges:
$$\sum_{n=1}^{+\infty} \frac{1}{n^p} \le 1+\frac{1}{p-1} < \infty.$$

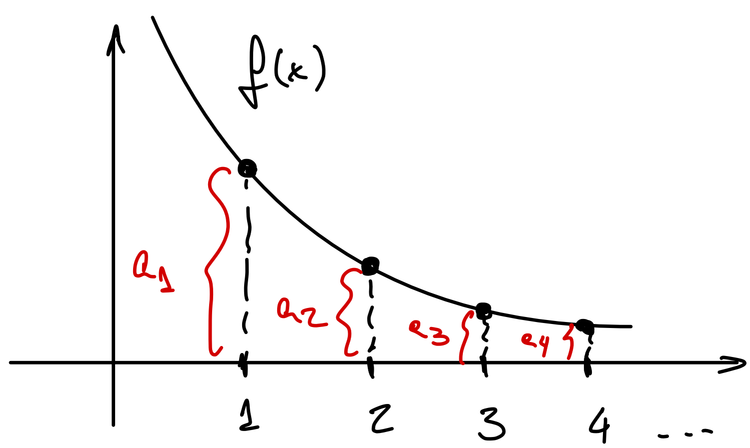
If $0<p\le 1$, then $\frac{1}{n^p} \ge \frac{1}{n}$. So by the Comparison Test,
$$\sum_{n=1}^{+\infty} \frac{1}{n^p} \ge \underbrace{\sum_{n=1}^{+\infty} \frac{1}{n}}_{\text{harmonic series}} = +\infty.$$

so $\displaystyle\sum_{n=1}^{+\infty} \frac{1}{n^p} = +\infty$.

Integral Test

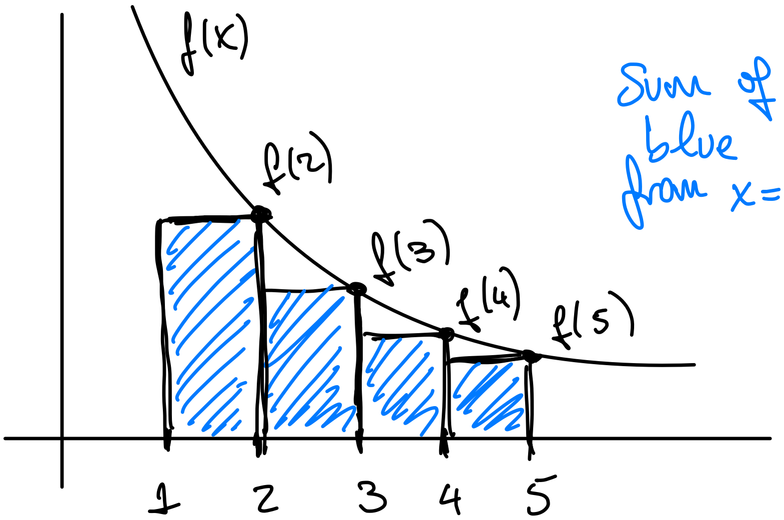
Suppose that $f(x)$ is continuous, positive, decreasing on the half-line $[1,+\infty)$. Then
$$\sum_{n=1}^{+\infty} a_n \text{ converges} \iff \int_1^{+\infty} f(x)\,dx \text{ converges}$$

where $a_n = f(n)$, $n\in\mathbb{N}$.



Pf: Consider the partial sums
$$s_n = \sum_{k=1}^{n} a_k = \sum_{k=1}^{n} f(k)$$

Since we are assuming $f(x)$ is positive, we have $f(k) > 0$, $\forall k \in \mathbb{N}$, so $(s_n)_{n\in\mathbb{N}}$ is an increasing sequence. Let us bound it from above using the assumption that $\int_1^{+\infty} f(x)\,dx < \infty$. For each $n \geq 2$:



Rectangles touching graph of $f(x)$ at the top right corner

Sum of areas of blue rectangles from $x=1$ up to $x=n$ $= f(2)+f(3)+\dots+f(n)$

$$= a_2 + a_3 + \dots + a_n$$

$$\leq \left(\text{Area under graph of } f(x) \text{ between } x=1 \text{ to } x=n.\right)$$

$$= \int_1^n f(x)\,dx.$$

Thus:

$$s_n = a_1 + a_2 + \dots + a_n$$

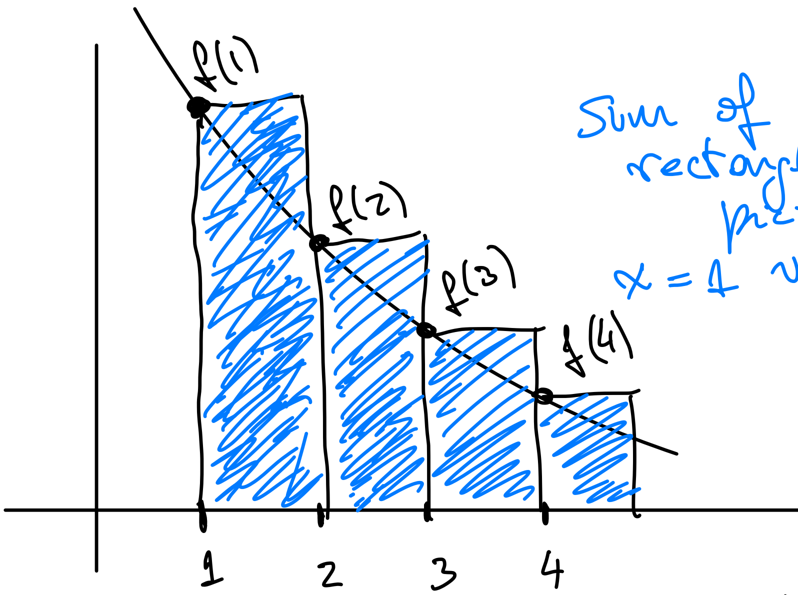
$$\leq a_1 + \int_1^n f(x)\,dx$$

Taking limits as $n \nearrow +\infty$,

$$\lim_{n\to\infty} s_n \leq \lim_{n\to\infty} a_1 + \int_1^n f(x)\,dx = a_1 + \int_1^{+\infty} f(x)\,dx < \infty$$

Therefore $(S_n)_{n\in\mathbb{N}}$ converges (b/c it is bounded from above and increasing).

Conversely, suppose $\sum_{n=1}^{+\infty} a_n$ converges; i.e. $(S_n)_{n\in\mathbb{N}}$ converges.



Sum of areas of rectangles in picture from $x=1$ up to $x=n+1$

$$= f(1) + f(2) + \cdots + f(n)$$

$$= a_1 + a_2 + \cdots + a_n$$

$$= S_n$$

$$\geq \left(\text{Area under graph of } f(x) \text{ from } x=1 \text{ up to } x=n+1.\right)$$

$$= \int_1^{n+1} f(x)\,dx.$$

Use rectangles that touch the graph of $f(x)$ at the top <u>left</u> corner

Thus, for all $n\in\mathbb{N}$,

$$S_n \geq \int_1^{n+1} f(x)\,dx$$

Taking limits as $n \nearrow +\infty$,

$$L = \lim_{n\to\infty} S_n \geq \lim_{n\to\infty} \int_1^{n+1} f(x)\,dx = \int_1^{+\infty} f(x)\,dx$$

hypothesis

Thus $\int_1^{+\infty} f(x)\,dx < \infty$.

$\square$

Alternating Series

signs alternate.

$$\sum_{n=1}^{+\infty} (-1)^{n+1} a_n = a_1 - a_2 + a_3 - a_4 + a_5 - a_6 + \dots$$

"Alternating series test"

Theorem. If $a_1 \geq a_2 \geq a_3 \geq \dots$, i.e., $(a_n)_{n\in\mathbb{N}}$ is decreasing and $\lim_{n\to\infty} a_n = 0$, then $\sum_{n=1}^{+\infty} (-1)^{n+1} a_n$ converges. Moreover $s_n := \sum_{k=1}^{n} (-1)^{k+1} a_k$ satisfy

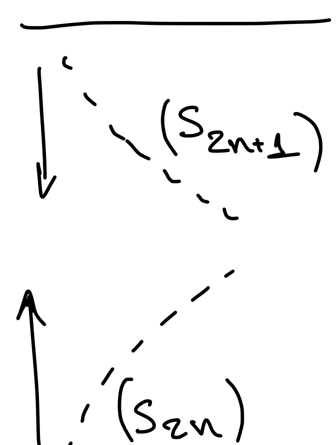
$$|s_n - s| \leq a_n \quad \text{for all } n \in \mathbb{N}, \text{ where } \lim_{n\to\infty} s_n = s.$$

Pf: Note that the subsequence $(s_{2n})_{n\in\mathbb{N}}$ is increasing:

$$s_{2n+2} - s_{2n} = -a_{2n+2} + a_{2n+1} \geq 0$$

and, similarly, the subsequence $(s_{2n+1})_{n\in\mathbb{N}}$ is decreasing:

$$s_{2n+1} - s_{2n-1} = a_{2n+1} - a_{2n} \leq 0.$$



All even elements are $\leq$ any odd element.

We claim that for all $n, m \in \mathbb{N}$

$$s_{2m} \leq s_{2n+1}.$$

First, note $s_{2n} \leq s_{2n+1}$ for all $n \in \mathbb{N}$ because $s_{2n+1} - s_{2n} = a_{2n+1} \geq 0$.

If $m \leq n$, then $S_{2m} \leq S_{2n} \leq S_{2n+1}$.

If $m \geq n$, then $S_{2n+1} \geq S_{2m+1} \geq S_{2m}$.

So $(S_{2m})_{m\in\mathbb{N}}$ is increasing and bounded from above by S_{2n+1} for any $n \in \mathbb{N}$, e.g., by S_3. Therefore $(S_{2m})_{m\in\mathbb{N}}$ converges.

Similarly $(S_{2n+1})_{n\in\mathbb{N}}$ is decreasing and bounded from below by S_{2m} for any $m\in\mathbb{N}$, e.g., S_2.

Thus, $(S_{2n+1})_{n\in\mathbb{N}}$ converges. Say

$$t = \lim_{n\to\infty} S_{2n+1}, \qquad S = \lim_{m\to\infty} S_{2m}.$$

Computing their difference, we find:

$$t - S = \lim_{n\to\infty} S_{2n+1} - \lim_{n\to\infty} S_{2n} = \lim_{n\to\infty} \underbrace{(S_{2n+1} - S_{2n})}_{a_{2n+1}}$$

$$= \lim_{n\to\infty} a_{2n+1} = 0.$$

Thus $t = S$, and hence $\lim_{n\to\infty} S_n = S$.

To prove $|S_n - S| \leq a_n$, note that for all $k\in\mathbb{N}$,

$$S_{2k} \le S \le S_{2k+1}$$

So:

$$S_{2k+1} - S \le S_{2k+1} - S_{2k} = a_{2k+1} \le a_{2k}$$

$$S - S_{2k} \le S_{2k+1} - S_{2k} = a_{2k+1} \le a_{2k}$$

So: $|S - S_n| \le a_n$ both if n is even or odd.

$\square$

<u>Examples:</u> Justify (using a convergence test) whether each of the following converges or diverges:

- $\displaystyle\sum_{n=1}^{+\infty} \frac{(-1)^{n+1}}{n} < \infty$ by the <u>Alternating Series Test</u>

  Alternating series w/ $a_n = \frac{1}{n}$ "Alternating harmonic Series"

  a_n decreasing, $\lim_{n\to\infty} a_n = 0$

- $\underbrace{\displaystyle\sum_{n=1}^{+\infty} \frac{5^n}{n!}}_{= e^5}$ <u>Ratio test</u> $a_n = \frac{5^n}{n!}$, $a_{n+1} = \frac{5^{n+1}}{(n+1)!}$

  $$\frac{a_{n+1}}{a_n} = \frac{5^{n+1}}{(n+1)!} \cdot \frac{n!}{5^n} = \frac{5}{n+1}$$

$$\limsup_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \frac{5}{n+1} = 0 < 1. \Rightarrow$$ Series converges <u>absolutely</u>

$$\sum_{n=0}^{+\infty}\left(\frac{2}{(-1)^n-3}\right)^n \quad \underline{\text{diverges}}.$$

$$|a_n|^{\frac{1}{n}} = \frac{2}{|(-1)^n-3|} = \begin{cases} \frac{2}{2} = 1 & n \text{ even} \\ \frac{2}{4} = \frac{1}{2} & n \text{ odd} \end{cases}$$

$\limsup_{n\to\infty} |a_n|^{\frac{1}{n}} = 1$ Root test <u>does not apply</u>

However, the above shows that $a_n = 1$ if n is even. So the sequence $(a_n)_{n\in\mathbb{N}}$ does not converge to 0, so $\sum_{n=1}^{+\infty} a_n$ diverges by the "n^{th}-term test".

- $$\frac{1}{2} + \frac{1}{3} + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{2^3} + \frac{1}{3^3} + \frac{1}{2^4} + \frac{1}{3^4} + \frac{1}{2^5} + \frac{1}{3^5} + \cdots$$

<u>Ratio test:</u>

$$\frac{a_{n+1}}{a_n} = \begin{cases} \dfrac{\left(\frac{1}{2}\right)^{m+1}}{\left(\frac{1}{3}\right)^m} = \frac{1}{2}\left(\frac{3}{2}\right)^m, & \text{if } n = 2m \text{ even}, \ m\in\mathbb{N} \\[2ex] \left(\frac{2}{3}\right)^m, & \text{if } n = 2m-1 \text{ odd}, \ m\in\mathbb{N} \end{cases}$$

$$\limsup_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{m\to\infty} \frac{1}{2}\left(\frac{3}{2}\right)^m = +\infty.$$

$$\liminf_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{m\to\infty} \left(\frac{2}{3}\right)^m = 0.$$

does not apply.

<u>Root test:</u>

$$|a_n|^{\frac{1}{n}} = \begin{cases} \left(\dfrac{1}{2^m}\right)^{\frac{1}{2m}} = \dfrac{1}{\sqrt{2}} & \text{if } n \text{ is odd} \\[2ex] \left(\dfrac{1}{3^m}\right)^{\frac{1}{2m}} = \dfrac{1}{\sqrt{3}} & \text{if } n \text{ is even.} \end{cases}$$

$$\limsup_{n\to\infty} |a_n|^{\frac{1}{n}} = \frac{1}{\sqrt{2}} < 1.$$

Root test applies and implies that the series converges absolutely.