Homework Set 2

DUE: SEP 27, 2021 (VIA BLACKBOARD, BY 11.59PM)

To be handed in:

Please remember that all problems will be graded!

1. [Ross, Exercises 4.1-4.4] Answer the following questions about each of the sets:

$$A = (-1, 1), \qquad B = \{\pi, e\}, \qquad C = \left\{\tfrac{1}{n} : n \in \mathbb{N} \text{ and } n \text{ is prime}\right\},$$
$$D = \{x^2 : x \in \mathbb{R}\}, \quad E = \bigcap_{n=1}^{\infty} \left[-\tfrac{1}{n}, 1 + \tfrac{1}{n}\right], \quad F = \left\{\sin\left(\tfrac{n\pi}{3}\right) : n \in \mathbb{N}\right\}.$$

   (i) Is it bounded from below? (If so, exhibit an explicit lower bound.)
   (ii) Is it bounded from above? (If so, exhibit an explicit upper bound.)
   (iii) Compute its infimum.
   (iv) Compute its supremum.

   Recall that $\inf S = -\infty$ *if* S *is unbounded from below; and* $\sup S = +\infty$ *if* S *is unbounded from above, as we chose to convention in Video 4 of Lecture 4.*

2. [Ross, Exercises 4.7(a), 5.6] Let $S, T \subset \mathbb{R}$ be nonempty subsets of $\mathbb{R}$, such that $S \subset T$. Prove that
$$\inf T \leq \inf S \leq \sup S \leq \sup T.$$
   Give concrete examples of sets S and T to show that some (which?) inequalities above might be equalities even if $S \subsetneq T$, i.e., even if S and T do not coincide.

Solutions

Text in blue represents side comments that are not integral parts of proofs, but address issues that some students might have had difficulties with in their attempted solutions.

1. $A = (-1, 1)$ is bounded from below by -1, bounded from above by $+1$. Moreover, $\inf A = -1$ and $\sup A = 1$;

   $B = \{\pi, e\}$ is bounded from below by e, bounded from above by π. Moreover, $\inf B = e$ and $\sup B = \pi$;

   $C = \left\{\frac{1}{n} : n \in \mathbb{N} \text{ and } n \text{ is prime}\right\}$ is bounded from below by 0, bounded from above by 1. Moreover, $\inf C = 0$ and $\sup C = \frac{1}{2}$;

   (Recall that 1 is not a prime number, so $1 \notin C$. The largest element in this set is $\max C = \sup C = \frac{1}{2}$. The assertion that $\inf C = 0$ follows from the Archimedean property together with the fact that there are *infinitely many prime numbers*, that is, the set of prime numbers is unbounded from above.)

$D = \{x^2 : x \in \mathbb{R}\}$ is bounded from below by 0 and not bounded from above. Moreover, $\inf D = 0$ and $\sup D = +\infty$;

$E = \bigcap_{n=1}^{\infty} \left[-\frac{1}{n}, 1 + \frac{1}{n} \right] = [0, 1]$ is bounded from below by 0 and bounded from above by 1. Moreover, $\inf E = 0$ and $\sup E = 1$;

$F = \left\{ \sin\left(\frac{n\pi}{3}\right) : n \in \mathbb{N} \right\} = \left\{ -\frac{\sqrt{3}}{2}, 0, \frac{\sqrt{3}}{2} \right\}$ is bounded from below by $-\frac{\sqrt{3}}{2}$ and bounded from above by $\frac{\sqrt{3}}{2}$. Moreover, $\inf F = -\frac{\sqrt{3}}{2}$ and $\sup F = \frac{\sqrt{3}}{2}$.

2. *Proof.* First, note that all the quantities in the desired chain of inequalities are finite if and only if T, and hence S, are bounded from below and from above. We may assume this is the case, since otherwise the inequalities become trivially true, because $-\infty \leq +\infty$ and $-\infty \leq x \leq +\infty$ for any real number $x \in \mathbb{R}$.

   Since $S \subset T$, *every* lower bound for T is also a lower bound for S. Indeed, if $\alpha \in \mathbb{R}$ is such that $\alpha \leq t$ for all $t \in T$, then also clearly $\alpha \leq s$ for all $s \in S$. In particular, the *largest* lower bound, $\inf T$, for the set T is also a lower bound for S. Since $\inf S$ is the largest lower bound for S, it follows that $\inf T \leq \inf S$. Analogously, every upper bound for T is an upper bound for S, and so is the *least* such upper bound, $\sup T$. Since $\sup S$ is the smallest among the upper bounds for S, it follows that $\sup S \leq \sup T$. The middle inequality is obvious, since $\inf S$ is a lower bound for S while $\sup S$ is an upper bound for S. $\square$

   Finally, each one of the above inequalities may (individually) be an equality even if the sets do not coincide; e.g., consider the intervals $S = (0, 1)$ and $T = [0, 1]$. Clearly, $\inf S = \inf T = 0$ and $\sup S = \sup T = 1$, but $S \neq T$, since $0 \in T$ but $0 \notin S$. The middle inequality can obviously be an equality without having $S = T$, e.g., take $S = \{1/2\}$, $T = [0, 1]$.